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The Characteristics of Systems which are Nearly in a State of Neutral Static Stability

-by-

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SUMMARY

It is shown that the rate of subsidence or divergence λ of a system which is near a state of neutral static stability can easly be calculated from a knowledge of the mode of displacement in the neutral state and this mode is found by solving a set of linear algebraic equations. The first order correction to the mode can also be found and in important cases this can be made the basis for calculating a second approximation to λ; if necessary a further correction to the mode can now be found and from this a still more accurate root can be calculated. The method can be extended to continuous systems having infinitely many degrees of freedom.

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CONTENTS

1. Introduction

Systems which are nearly in a state of neutral static stability are of considerable interest and technical importance. Exactly neutral systems are specially simple in as much as the neutral mode of displacement can be found merely by solving a set of simultaneous linear algebraic equations. It is shown in this paper that the small rate of divergence or subsidence λ of a nearly neutral system can be very easily calculated to the first order of small quantities from a knowledge of the neutral mode; the first order corrections to this mode can readily be found also. When the matrices of the inertias, dampings and stiffnesses are symmetric the mode correct to the first order can be made the basis of a calculation of λ correct to the second order. For a continuous system with infinitely many freedoms the first order approximation to λ can be derived from the values of the potential energy and rate of dissipation of energy in an imaginary divergent motion proportional to the neutral mode and having λ equal to unity.

2. Exactly Neutral Systems

The matrices of the inertias, damping coefficients and stiffnesses are, as usual, denoted by A, B and C respectively. These are square matrices of order n where n is the number of degrees of freedom of the system. The static displacements q corresponding to static loads Q are given by

$$Cq = Q \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2,1)$$

$$\text{or} \quad q = C^{-1}Q. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2,2)$$

If the determinant of C becomes very small the elements of C^{-1} become very large and the displacements q corresponding to given loads Q also become very large. When $|C|$ vanishes the system is in a state of neutral static stability and then finite displacements can occur in the absence of applied loads.* Since the constant term in the determinantal equation of the system is equal to $|C|$, the equation has the root $\lambda = 0$ when $|C|$ vanishes. Thus the

/velocities ...

* We can suppose this motion started by an extremely small impulse.

velocities $\dot{q}$ and accelerations $\ddot{q}$ are zero in the mode of displacement corresponding to the root $\lambda = 0$. There are, however, (n-1) other modes which, in general, correspond to non-vanishing roots λ and in these modes the velocities and accelerations are not, in general, zero. Cases of multiple zero roots can arise but are rare. When a displacement occurs in the neutral mode the forces are always balanced. Hence the total work done in a neutral displacement is zero. As an example, when the angle of incidence of an aircraft varies slowly when the centre of mass is at the longitudinal neutral point the resultant aerodynamic force is constantly equal to the weight of the aircraft.

3. Nearly Neutral Systems

Suppose next that the system is near but not in the state of neutral static stability. There are then some simple methods for finding the characteristics of the nearly neutral mode.

(a) The matrices A, B and C are all symmetric

In this instance we can apply an extension of Rayleigh's Principle. Suppose that in a datum state of the system there is a characteristic root λ and a corresponding modal column q. Then λ must satisfy exactly the quadratic equation

$$\lambda^2(q'Aq) + \lambda\,(q'Bq) + q'C\,q = 0 \qquad \ldots\ldots\ldots\ldots(3,1)$$

where the coefficients are quadratic forms in the elements of q. This equation is obtained from the dynamical equation in matrix form

$$\lambda^2\,Aq + \lambda Bq + Cq = 0$$

by premultiplication by q', the transposed of q, i.e. the row matrix having the same elements as the column q; the equation is true whether A, B and C are symmetric or not. Now let A, B and C be symmetric and let the elements of q depart from their true values by small quantities of the first order. Then according to the principle the value of the root λ obtained from the quadratic equation (3,1) will only be in error by a small quantity of the second order*. Accordingly, let A, B and C refer to the near neutral state and

/substitute ...

* See p.300 of 'Elementary Matrices' by R.A. Frazer, W.J. Duncan and A.R. Collar (Cambridge, 1938).

substitute for q the neutral modal column q_o (which is very easy to calculate). The root will be a small quantity of the first order and λ^2 will be of the same order as the terms neglected and therefore to be discarded. Hence we derive the approximation

$$\lambda = - \frac{q_o' \, C \, q_o}{q_o' \, B \, q_o} \qquad \text{..............(3,2)}$$

which is easy to compute.
Let C reduce to C_o in the exactly neutral state so that

$$C_o \, q_o = 0. \qquad \text{...............(3,3)}$$

Also let $C = C_o + \delta C$ ....(3,4)

where the elements of δC are small quantities of the first order and C_o is symmetric. Then by (3,3) and (3,4)

$$q_o' \, C \, q_o = q_o' \, \delta C \, q_o \qquad \text{...............(3,5)}$$

and (3,2) becomes

$$\lambda = - \frac{q_o' \, \delta C \, q_o}{q_o' \, B \, q_o}, \qquad \text{...............(3,6)}$$

which is correct to the first order of small quantities. From knowledge of this approximation to λ we can calculate the first order correction to the modal column q_o merely by solving a set of simultaneous linear equations (see equation (3,13)).

The approximation to λ can now be improved by use of the corrected modal column and the equation (3,1). Since λ is a small quantity of the first order equation (3,1) will be correct to the second order if $q'Bq$ is correct to the first order and $q'Cq$ to the second order. Now suppose that all the elements of δC are proportional to p, which is thus of the first order, and let

$$\delta C = p \, C_1. \qquad \text{..............(3,7)}$$

Suppose also that to the second order

$$q = q_o + pq_1 + p^2 q_2, \qquad \text{..............(3,8)}$$

/where ...

where q_1 is now known but q_2 is not. Then to the second order of small quantities

$$q'Cq = p\,(q_o' C_o q_1 + q_o' C_1 q_o)$$

$$+ p^2\,(q_o' C_o q_2 + q_o' C_1 q_1 + q_1' C_o q_1 + q_1' C_1 q_o) \qquad \ldots\ldots\ldots\ldots\ldots(3,9)$$

in view of (3,3). But equation (3,3) gives by transposition

$$q_o' C_o = 0 \qquad \ldots\ldots\ldots\ldots\ldots(3,10)$$

since C_o is symmetric. Hence (3,9) becomes

$$q' Cq = p\, q_o' C_1 q_o$$

$$+ p^2\,(q_o' C_1 q_1 + q_1' C_o q_1 + q_1' C_1 q_o) \ldots.(3,11)$$

Now this does not contain q_2 and we conclude that equation (3,1) will be correct to the second order of small quantities when we substitute for q the approximate modal column which is correct to the first order. Hence the small root of (3,1) will now be correct to the second order.

The improved approximation to λ obtained from the quadratic equation (3,1) can be used to obtain an improved approximation to the corresponding mode by use of the dynamical equations. Then this recalculated mode may, if desired, be used to obtain a still better approximation to λ by means of (3,1). The whole process can be repeated as often as desired.

(b) <u>The matrices A, B and C are general, except that C is nearly singular</u>

The dynamical equations in matrix form yield

$$(\lambda^2 A + \lambda B + C_o + \delta C)(q_o + \delta q) = 0 \qquad \ldots\ldots\ldots\ldots(3,12)$$

while (3,3) is again satisfied. If we now neglect second order terms we obtain

$$\lambda B q_o + C_o \delta q = - \delta C q_o. \qquad \ldots\ldots\ldots\ldots(3,13)$$

/This ...

This represents n linear scalar equations and there are apparently $(n + 1)$ unknowns namely λ and the n elements of δq. However, the modal column is arbitrary to a scalar multiplier and we may accordingly assign one of the elements of δq, say make one element zero. We have now n unknowns, which can be calculated.

We shall now show that when C_o is symmetric this procedure leads exactly to the value of λ given by (3,2) or (3,6). Premultiply (3,13) by q_o'. We obtain

$$\lambda q_o' B q_o = - q_o' (C_o \delta q + \delta C q_o)$$

$$= - q_o' \delta C q_o$$

by (3,10) and this is identical with (3,6).

4. A Simple Numerical Example

The inertia matrix is

$$A = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4,1)$$

The damping matrix is

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4,2)$$

The stiffness matrix is

$$C = C_o + \delta C \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4,3)$$

where

$$C_o = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4,4)$$

and is singular, while

$$\delta C = p \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4,5)$$

where p is supposed to be small. The matrices are all symmetric so both methods described above are applicable. To facilitate

/Comparisons ...

comparisons with the approximations which we shall obtain we give the expansion of the exact determinantal equation for λ , namely

$$\begin{aligned} \left| \lambda^2 A + \lambda B + C \right| \\ \equiv 83\lambda^6 + 96\lambda^5 + (16p + 144)\lambda^4 \\ + (14p + 86)\lambda^3 + (21p + 33)\lambda^2 \\ + (8p + 6)\lambda + p = 0. \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots (4,6)$$

When p is zero this has a zero root in accordance with C_o being singular. The modal column q_o corresponding to the zero root is obtained at once from the linear equation

$$C_o \; q_o = 0$$

and is found to be

$$q_o = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots (4,7)$$

where the last element has been assigned the value unity. When p is given the value 0.1 we find from (4, 6) the true value of the numerically small root and the corresponding modal column to be

$$\lambda = -0.01597_1 \qquad \ldots\ldots\ldots\ldots\ldots (4,8)$$

$$q = \begin{bmatrix} -2.22535 \\ 1.13840 \\ 1.0 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots (4,9)$$

to 5 places of decimals.

The first approximation to the small root as given by (3, 6) is

$$\lambda = -\frac{p}{6} = -0.01667. \qquad \ldots\ldots\ldots\ldots\ldots (4,10)$$

Equation (3, 13) yields the same value of λ and when we assign the zero value to the final element in δq we obtain the approximate modal column

$$\begin{bmatrix} -2.21667 \\ 1.13333 \\ 1.0 \end{bmatrix}. \qquad \ldots\ldots\ldots\ldots\ldots (4,11)$$

/On ...

On substitution of this value of q in equation (3,1) we obtain the quadratic equation

$$22.12778\lambda^2 + 7.20500\lambda$$

$$+ 0.10944_4 = 0 \qquad \text{..............} (4,12)$$

of whichnthe smaller root is

$$\lambda = - 0.01597_4. \qquad \text{..............} (4,13)$$

This differs from the true root only by about 1 part in 5,000. The error in the modal column is of the order of $\frac{1}{2}$ per cent and the departure from the neutral modal column is about 10 per cent.

For comparison, the calculation has been repeated with the inertia matrix reduced to one quarter of that given in (4,1), i.e. with all the inertia coefficients reduced to one quarter of their former values. The first order approximations to λ and to the mode remain as in (4,10) and (4,11) respectively. However, the quadratic equation (3,1) giving the second approximation to λ is obtained from (4,12) by reducing the coefficient of λ^2 to one quarter of the value in (4,12) and the second approximation is

$$\lambda = - 0.01537_1. \qquad \text{..............} (4,14).$$

Now the 'exact' root as obtained from the determinantal equation is

$$\lambda = - 0.01537_0 \qquad \text{..............} (4,15)$$

so the error is only about 1 part in 15,000 or say one third of the former error. The 'exact' mode is

$$q = \begin{bmatrix} -2.22377 \\ 1.13745 \\ 1.0 \end{bmatrix} \qquad \text{..............} (4,16)$$

so the first approximation to the mode is in error by roughly 0.3 per cent.

5. Details of the Procedure in Dealing with a Nearly Neutral System

We suppose that the determinant $|C|$ of the stiffness matrix has been evaluated and found to be small, so the system is near a state of neutral static stability. It is now convenient to consider an artificial state of the system which is exactly neutral and we can derive this merely by varying one element of C; this element may be chosen so as to simplify the calculation. The increment of the chosen element is $-|C|/K$ where K is its cofactor. We next derive, merely by solving a set of linear equations, the neutral mode q_o and may then proceed as explained in §3. There is some advantage in choosing the element of the modal column which is arbitrarily kept fixed so that the product of this element and the varied element of C occurs in the calculations.

6. Treatment of Continuous Systems

The methods given above are not, in general, immediately applicable to continuous systems having infinitely many degrees of freedom. We may, however, treat such systems with any desired degree of approximation by the 'Lagrangian' method in which the continuous system is represented by another having a finite number of degrees of freedom*. This method, also known as 'the method of semi-rigid representation', has been much used in dealing with aero-elastic problems and is of great utility. The methods explained in the present paper can be applied at once to the semi-rigid representation of the sytem.

Another method can be applied directly when the forces dependent on the displacements are derivable from a single-valued potential, for then the stiffness matrix is symmetric. We saw in §3 under item (b) that equation (3,2) is valid to the first order of small quantities when C_o is symmetric whatever may be the nature of A and B. Now C_o will be symmetric in the circumstances we have postulated so (3,2) is applicable however many degrees of freedom the system may possess. To assist the application of the formula we now give a physical interpretation of the numerator and denominator. We know that the potential energy stored in the system when the displacement is q_o is

$$V = \tfrac{1}{2}\, q_o' \, C \, q_o. \qquad \text{.................(6,1)}$$

Next, the time rate of dissipation of energy in the system is

$$D = \dot{q}' \, B\dot{q} \qquad \text{.................(6,2)}$$

/ and ...

* See, for example, Chap. V of 'Mechanical Admittances and their Applications to Oscillation Problems' by W.J. Duncan. R. and M. 2000 (1946).

and if we suppose that

$$q = q_0 e^{\lambda t} \qquad \text{................} (6,3)$$

this becomes

$$D(\lambda) = \lambda^2 q_0' B q_0 . \qquad \text{................} (6,4)$$

Thus we see that

$$q_0' B q_0 = D(1) \qquad \text{................} (6,5)$$

where $D(1)$ is the rate of dissipation of energy when

$$q = q_0 e^t . \qquad \text{................} (6,6)$$

Accordingly equation (3,2) becomes

$$\lambda = - \frac{2V}{D(1)} . \qquad \text{.................} (6,7)$$

The numerator and denominator can be calculated for a continuous system when the neutral mode of displacement is known

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