REPORT No.18

July, 1948

THE COLLEGE OF AERONAUTICS

CRANFIELD

On the Integration of Hyperbolic Differential Equations

-by-

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SUMMARY

A new method is developed for the analysis and integration of linear partial differential equations of two independent variables and of arbitrary order,

$$\sum_{k=0}^{n} \sum_{\ell=0}^{k} a_{k\ell}(x,y) \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} = a_0(x,y)$$

for given initial conditions. It is shown that this equation can be replaced by systems of equations of the form

$$\frac{\partial f_i}{\partial x} + c_i \frac{\partial f_i}{\partial y} = F_i(x,y,f_1,f_2, \dots , fm) \ , \ i = 1,2,\dots m$$

where the $f_i = f_i(x,y)$ are the dependent variables. The expression on the left hand side represents the derivative of f_i with respect to x along one of the characteristic curves of the system. Two methods of integration which are based on this fact are put forward, a step-by-step method, and a method of successive approximation. A detailed convergence proof is given for the latter. It is suggested that the methods can be adapted to numerical calculations.

DMS.

1. Introduction

In the present paper, we shall be concerned with the integration of linear partial differential equations of two independent variables and of arbitrary order, viz.,

$$\sum_{k=0}^{n} \sum_{\ell=0}^{k} a_{k\ell} \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} = a_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where x and y are the two independent variables, z is the dependent variable, n is a positive integer and the $a_{k\ell}$ and a_0 are functions of x and y.

The parametric representation $x = x(t)$, $y = y(t)$ of the characteristic curves associated with (1) is given by the differential equation

$$\begin{vmatrix} a_{n0} & a_{n1} & a_{n2} & \cdots\cdots & a_{n,n-1} & a_{nn} \\ \dot{x} & \dot{y} & & & & \cdot \\ & \dot{x} & \dot{y} & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & \dot{x} & \dot{y} \end{vmatrix} = 0$$

i.e.,

$$a_{n0}\,\dot{y}^n - a_{n1}\,\dot{y}^{n-1}\,\dot{x} + \ldots + (-1)^{n-1}\,a_{n,n-1}\,\dot{y}\dot{x}^{n-1} + (-1)^n\,a_{nn}\dot{x}^n = 0 \quad \ldots\ldots\ldots(2)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$. Putting $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \gamma$, equation (2) may also be written as the 'characteristic equation'

$$a_{n0}\gamma^n - a_{n1}\,\gamma^{n-1} + \ldots + (-1)^{n-1}\,a_{n,n-1}\gamma + (-1)^n\,a_{nn} = 0 \quad \ldots(3)$$

The differential equation (1) is said to be hyperbolic in a given region of the (x,y) plane if the roots of (3) are all real and distinct for all points of the region. In the sequel, we shall be concerned exclusively with the hyperbolic case. The significance of the characteristic curves - which can be defined by various properties, e.g., as potential carriers of discontinuities of the $n^{th.}$ derivatives - will appear clearly in the present analysis.

Our main object will be the development of a new method for the integration of a hyperbolic equation of type (1) for given initial conditions. We shall confine ourselves to the standard case in which the function z and its first $(n-1)$ derivatives with respect to x are given over a range of values of y for $x = 0$. For analytical coefficients and analytical initial conditions, the existence of a solution is ensured by the fundamental

/theorem ....

theorem of Cauchy-Kowalewski. However, the requirement of analyticity is unnaturally stringent for problems of this type. More recently, work by K.Friedrichs and H.Lewy (ref.1) has established the existence and uniqueness of the solution under weaker conditions. (The paper referred to goes further than this as it deals with the general non-linear equation of two independent variables). F. Rellich (ref.2) has generalised Riemann's method to cope with essentially the same problem as considered in the present note. The case of constant coefficients had been treated earlier by G. Herglotz by a Fourier integral method (ref.3).

The present method establishes the existence and uniqueness of the solution under considerably less stringent conditions than postulated hitherto. Also, it leads to procedures which, with some modifications, should be suitable for numerical purposes. It can moreover be adapted to deal with some cases - discontinuity of coefficients, etc., - which may be important for the applications but which are outside the scope of other methods. However, in the present paper we shall confine ourselves to the solution of the standard problem mentioned above. The central idea of the method first arose in connection with some work on stress propagation in beams (refs. 4, 5).

To introduce this idea, we consider the case of a vibrating string which has inspired so many other theories of partial differential equations,

$$\frac{\partial^2 z}{\partial x^2} - c^2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

with the initial conditions $z = \phi(y)$, $\frac{\partial z}{\partial x} = \psi(y)$ for $x = 0$.

In this equation, x denotes the time, y the coordinate parallel to the undisturbed string, and z its deflection. The coefficient c^2 $(c > 0)$ which, in general, is a function of y, is given by the ratio of the tension in the string and of its linear density. We shall make the conventional assumption that the string is infinite in both directions.

Assume first that c is constant. Then the general solution of (3) is given by

$$z = f(y - cx) + g(y + cx) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and the functions f and g can be determined from the boundary conditions.

However, we may proceed in a different manner which, though considerably more complicated in the present case, can be extended to the general problem (1) where a functional solution is no longer available.

For this purpose we put

$$\begin{aligned} f_1(x,y) &= \frac{\partial z}{\partial x} - c\frac{\partial z}{\partial y} \\ f_2(x,y) &= \frac{\partial z}{\partial x} + c\frac{\partial z}{\partial y} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

/Then, ...

Then, by (3),

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x} + c\,\frac{\partial f_1}{\partial y} &= 0 \\ \frac{\partial f_2}{\partial x} - c\,\frac{\partial f_2}{\partial y} &= 0 \end{aligned}\right\} \quad \ldots\ldots(6)$$

Equation (6) shows that f_1 and f_2 are propagated along the string without distortion, with velocities c and $-c$ respectively, $f_1(x,y) = h_1(y - cx)$, $f_2(x,y) = h_2\,(y + cx)$, say. The initial conditions for f_1 and f_2 are obtained from the initial conditions for z, viz.,

$$\left.\begin{aligned} f_1(0,y) &= \psi(y) - c\phi'(y) \\ f_2(0,y) &= \psi(y) + c\phi'(y) \end{aligned}\right\} \quad \ldots\ldots(7)$$

Hence

$$\left.\begin{aligned} f_1(x,y) &= \psi(y - cx) - c\phi'(y - cx) \\ f_2(x,y) &= \psi(y + cx) + c\phi'(y + cx) \end{aligned}\right\} \quad \ldots\ldots(8)$$

Having determined $f_1(x,y)$ and $f_2(x,y)$, we may now find $z(x,y)$ by solving either of the two first order equations (5) for z. Now the solution of the equation

$$\frac{\partial z}{\partial x} - c\,\frac{\partial z}{\partial y} = F(x,y) \quad \ldots\ldots(9)$$

is

$$z(x,y) = z(0,y + cx) + \int_0^x F(\xi, y + c(x - \xi))\,d\xi \quad \ldots\ldots(10)$$

Hence, from the first of the two equations (5), we have, taking into account (8),

$$\begin{aligned} z(x,y) &= \phi(y + cx) + \int_0^x \Big[\psi(y + c(x - \xi) - c\,\xi) - c\phi'(y + c(x - \xi) \\ &\qquad - c\,\xi)\Big]\,d\xi \\ &= \phi(y + cx) + \int_0^x \Big[\psi(y + cx - 2c\xi) - c\phi'(y + cx - 2c\,\xi)\Big]\,d\xi \end{aligned}$$

Substituting $\eta = y + cx - c\,\xi$ as variable of integration, we obtain

$$z(x,y) = \phi(y + cx) + \frac{1}{2c}\int_{y-cx}^{y+cx} \Big[\psi(\eta) - c\phi'(\eta)\Big]\,d\eta$$

and so

$$z(x,y) = \frac{1}{2}\Big[\phi(y + cx) + \phi(y - cx)\Big] + \frac{1}{2c}\int_{y-cx}^{y+cx} \psi(\eta)\,d\eta \quad \ldots\ldots(11)$$

/in .....

in agreement with the well known result obtained more directly by the use of (4).

Assume now that the density of the string is variable so that c is a function of y. Defining $f_1(x,y)$ and $f_2(x,y)$ by (5) as before, we now have

$$\frac{\partial f_1}{\partial x} + c\frac{\partial f_1}{\partial y} = \frac{\partial^2 z}{\partial x^2} - c^2 \frac{\partial^2 z}{\partial y^2} - c\frac{dc}{dy}\frac{\partial z}{\partial y} = -c\frac{dc}{dy}\frac{\partial z}{\partial y}$$

and similarly

$$\frac{\partial f_2}{\partial x} - c\frac{\partial f_2}{\partial y} = -c\frac{dc}{dy}\frac{\partial z}{\partial y}.$$

Now

$$\frac{\partial z}{\partial y} = \frac{1}{2c}\left(f_2 - f_1\right)$$

and so

$$\begin{aligned}\frac{\partial f_1}{\partial x} + c\frac{\partial f_1}{\partial y} &= \frac{1}{2}\frac{dc}{dy}\left(f_1 - f_2\right)\\ \frac{\partial f_2}{\partial x} - c\frac{\partial f_2}{\partial y} &= \frac{1}{2}\frac{dc}{dy}\left(f_1 - f_2\right)\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Now the left hand sides of (12) denote differentiation with respect to x along the characteristic curves of the equation (3), ($y \pm cx = \text{const.}$). Therefore the right hand sides of (12) measure the distortion of the "waves" f_1 and f_2 as they travel. They are the mathematical expression for the exchange of energy and momentum between the two waves, which is due to reflection.

Having solved (12), we may then regard either of the two equations in (5) as an equation for $z(x,y)$ and in this way determine the latter function.

The fundamental idea of the present work is to reduce the general problem (1) to the consideration of a system of "waves" travelling at different speeds and continuously affecting each other in the course of their propagation. This idea will now be expressed in formal language.

We have seen how the integration of the equation of the vibrating string can be reduced to the successive solution of a system of two linear partial differential equations of the first order and of a single equation of the same type. It will be shown in section 2 of the present paper that the integration of a hyperbolic equation of type (1) can always be reduced to the successive integration of systems of linear equations of the type

$$\frac{\partial f_i}{\partial x} + c_i\frac{\partial f_i}{\partial y} = b_{i1} f_1 + b_{i2} f_2 + \ldots + b_{im} f_m + b_i \;,$$

$$i = 1, 2, \ldots, m \quad m \leq n \quad \ldots(13)$$

/where ..

where the coefficients c_i, b_i and b_{ik} are functions of x and y which depend on the coefficients of the differential equation (1), but not on the initial conditions. However, in the process of determining the coefficients b_{ik}, we shall have to find particular solutions of systems of partial differential equations of the more general type

$$\frac{\partial f_i}{\partial x} + c_i \frac{\partial f_i}{\partial y} = F_i(x,y,f_1,f_2, \ldots , fm) \ , \ i = 1,2, \ldots , m \quad ..(14)$$

The central feature of both (13) and (14) is that their left hand sides represent the derivatives of the unknown functions along the characteristic curves of the system, while the right hand sides only involve the unknown functions but not their derivatives. A method of integration for systems of this type is developed in section 3.

2. The replacement of an equation of order n by special systems of first order equations

We proceed to establish a system of equations of type (13) which is related to the given differential equation

$$\sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} = a_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

on the assumption that the roots of

$$a_{n0} \gamma^n - a_{n1} \gamma^{n-1} + \ldots + (-1)^{n-1} a_{n,n-1} \gamma + (-1)^n a_{nn} = 0 \quad \ldots\ldots\ldots(3)$$

are all real and distinct in a given region of the (x,y) plane. We may then assume that $a_{n0} = 1$. We shall suppose in the first instance only that the coefficients $a_{k\ell}$ and a_0 possess continuous derivatives of the first order. Then the roots γ_k, $k=1,2, \ldots , k$ of (3) are continuous and distinct functions of x and y in the region in question.

It will be seen that the roots of the equation

$$\gamma^n + a_{n1} \gamma^{n-1} + \ldots + a_{n,n-1} \gamma + a_{nn} = 0$$

are $\quad -\gamma_1, -\gamma_2, \ldots, -\gamma_n$. We may therefore write

$$\gamma^n + a_{n1}\gamma^{n-1} + \ldots + a_{n,n-1}\gamma + a_{nn} =$$

$$= \left(\gamma + \gamma_m\right)\left(\gamma^{n-1} + a_1^{(m)} \gamma^{n-2} + \ldots + a_{n-1}^{(m)}\right), \ m = 1,2, \ldots n \quad \ldots\ldots\ldots(15)$$

where the coefficients $a_\ell^{(m)}$ are functions of x and y.

/Then ....

Then

$$\left(\frac{\partial}{\partial x}+\gamma_m\frac{\partial}{\partial y}\right)\left(\frac{\partial^{n-1}z}{\partial x^{n-1}}+a_1^{(m)}\frac{\partial^{n-1}z}{\partial x^{n-2}\partial y}+a_2^{(m)}\frac{\partial^{n-1}z}{\partial x^{n-3}\partial y^2}+\dots+a_{n-1}^{(m)}\frac{\partial^{n-1}z}{\partial y^{n-1}}\right)=$$

$$=\frac{\partial^n z}{\partial x^n}+a_{n1}\frac{\partial^n z}{\partial x^{n-1}\partial y}+a_{n2}\frac{\partial^n z}{\partial x^{n-2}\partial y^2}+\dots+a_{nn}\frac{\partial^n z}{\partial y^n}+$$

$$+\left(\frac{\partial a_1^{(m)}}{\partial x}+\gamma_m\frac{\partial a_1^{(m)}}{\partial y}\right)\frac{\partial^{n-1}z}{\partial x^{n-2}\partial y}+\left(\frac{\partial a_2^{(m)}}{\partial x}+\gamma_m\frac{\partial a_2^{(m)}}{\partial y}\right)\frac{\partial^{n-1}z}{\partial x^{n-3}\partial y^2}$$

$$+\dots+\left(\frac{\partial a_{n-1}^{(m)}}{\partial x}+\gamma_m\frac{\partial a_{n-1}^{(m)}}{\partial y}\right)\frac{\partial^{n-1}z}{\partial y^{n-1}}\qquad\dots\dots\dots\dots(16).$$

Hence, putting

$$z_m(x,y)=\frac{\partial^{n-1}z}{\partial x^{n-1}}+a_1^{(m)}\frac{\partial^{n-1}z}{\partial x^{n-2}\partial y}+\dots+a_{n-1}^{(m)}\frac{\partial^{n-1}z}{\partial y^{n-1}},$$

$$m=1,2,\dots,m\qquad\dots\dots\dots\dots\dots(17)$$

we obtain from (1)

$$\frac{\partial z_m}{\partial x}+\gamma_m\frac{\partial z_m}{\partial y}=a_0-\sum_{k=0}^{n-1}\sum_{\ell=0}^{k}a_{k\ell}\frac{\partial^k z}{\partial x^{k-\ell}\partial y^\ell}+$$

$$+\left(\frac{\partial a_1^{(m)}}{\partial x}+\gamma_m\frac{\partial a_1^{(m)}}{\partial y}\right)\frac{\partial^{n-1}z}{\partial x^{n-2}\partial y}+\dots+\left(\frac{\partial a_{n-1}^{(m)}}{\partial y}+\gamma_m\frac{\partial a_{n-1}^{(m)}}{\partial y}\right)\times$$

$$\times\frac{\partial^{n-1}z}{\partial y^{n-1}}\quad m=1,2,\dots,n\qquad\dots\dots\dots\dots(18)$$

We now define functions $f_m(x,y)$, $m = 1, 2, \dots, n$, by

$$f_m(x,y)=z_m(x,y)+\sum_{k=0}^{n-2}\sum_{\ell=0}^{k}b_{k\ell}^{(m)}\frac{\partial^k z}{\partial x^{k-\ell}\partial y^\ell}\qquad\dots\dots\dots\dots(19)$$

where the coefficients $b_{k\ell}^{(m)} = b_{k\ell}^{(m)}(x,y)$ will be determined presently.

/From ....

From (17) and (19)

$$f_m(x,y) = \frac{\partial^{n-1} z}{\partial x^{n-1}} + a_1^{(m)} \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \dots + a_{n-1}^{(m)} \frac{\partial^{n-1} z}{\partial y^{n-1}}$$

$$+ \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b_{k\ell}^{(m)} \frac{\partial^n z}{\partial x^{k-\ell}\partial y^{\ell}} \quad m = 1,2,..,n \dots$$

$$\dots\dots\dots\dots(20)$$

or

$$\frac{\partial^{n-1} z}{\partial x^{n-1}} + a_1^{(m)} \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \dots + a_{n-1}^{(m)} \frac{\partial^{n-1} z}{\partial y^{n-1}} = f_m(x,y)$$

$$- \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b_{k\ell}^{(m)} \frac{\partial^k z}{\partial x^{k-\ell}\partial y^{\ell}}$$

$$m = 1,2, \dots, n \dots\dots\dots\dots\dots\dots\dots(21)$$

It will be shown below that the determinant of (20), looked upon as a system of equations for the derivatives

$$\frac{\partial^{n-1} z}{\partial x^{n-1}}, \ \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}, \ \dots, \ \frac{\partial^{n-1} z}{\partial y^{n-1}} \quad ,$$

does not vanish. Writing $\left[c_{im}\right]$ for the inverse of the matrix

$$\left[1, a_1^{(m)}, a_2^{(m)}, \dots, a_{n-1}^{(m)}\right] \ ; \quad m = 1,2, \dots, n,$$

we then have

$$\frac{\partial^{n-1} z}{\partial x^{n-1-i}\partial y^i} = \sum_{m=1}^{n} c_{im} \left(f_m(x,y) - \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b_{k\ell}^{(m)} \frac{\partial^k z}{\partial x^{k-\ell}\partial y^{\ell}} \right)$$

$$i = 0, 1, \dots, n-1 \dots\dots\dots\dots(22).$$

Also, from (18)

$$\frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} = a_0 - \sum_{k=0}^{n-2} \sum_{i=0}^{k} a_{ki} \frac{\partial^k z}{\partial x^{k-i}\partial y^i}$$

$$+ \sum_{i=0}^{n-1} \left(\frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i} \right) \frac{\partial^{n-1} z}{\partial x^{n-1-i}\partial y^i}$$

$$m = 1,2, \dots, n \dots\dots\dots(23)..$$

where we define the functions $a_0^{(m)}(x,y)$ which occur here for the first time for convenience by $a_0^{(m)}(x,y) = 0$,

Replacing the derivatives $\dfrac{\partial^{n-1}z}{\partial x^{n-1-i}\partial y^i}$ in (23) by the expressions given in (22),

$$\frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} = a_0 - \sum_{k=0}^{n-2}\sum_{i=0}^{k} a_{ki}\frac{\partial^k z}{\partial x^{k-i}\partial y^i}$$

$$+ \sum_{i=0}^{n-1}\sum_{p=0}^{n}\left(\frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i}\right) c_{ip}\left(f_p(x,y)\right.$$

$$\left. - \sum_{k=0}^{n-2}\sum_{\ell=0}^{k} b_{k\ell}^{(p)} \frac{\partial^k z}{\partial x^{k-\ell} y^\ell}\right) \qquad \ldots\ldots\ldots(24)$$

Now, from (19),

$$\frac{\partial f_m}{\partial x} + \gamma_m \frac{\partial f_m}{\partial y} = \frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} + \left(\frac{\partial}{\partial x} + \gamma_m \frac{\partial}{\partial y}\right)\left(\sum_{k=0}^{n-2}\sum_{\ell=0}^{k} b_{k\ell}^{(m)} \frac{\partial^k z}{\partial x^{k-\ell}\partial y^\ell}\right)$$

$$= \frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} + \sum_{k=0}^{n-2}\sum_{\ell=0}^{k}\left[\left(\frac{\partial b_{k\ell}^{(m)}}{\partial x} + \gamma_m \frac{\partial b_{k\ell}^{(m)}}{\partial y}\right)\frac{\partial^k z}{\partial x^{k-\ell}\partial y^\ell}\right.$$

$$\left. + b_{k\ell}^{(m)}\left(\frac{\partial^{k+1} z}{\partial x^{k+1-\ell}\partial y^\ell} + \gamma_m \frac{\partial^{k+1} z}{\partial x^{k-\ell}\partial y^{\ell+1}}\right)\right]$$

$$= \frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} + \sum_{k=0}^{n-1}\sum_{\ell=0}^{k}\left[\left(\frac{\partial b_{k\ell}^{(m)}}{\partial x} + \gamma_m \frac{\partial b_{k\ell}^{(m)}}{\partial y}\right)\right.$$

$$\left. + \left(b_{k-1,\ell}^{(m)} + \gamma_m b_{k-1,\ell-1}^{(m)}\right)\right]\frac{\partial^k z}{\partial x^{k-\ell}\partial y^\ell} \qquad \ldots\ldots\ldots\ldots(25)$$

where we have put $b_{-1,\ell}^{(m)} = b_{n-1,\ell}^{(m)} = b_{k,k+1}^{(m)} = 0$, $m = 1, 2, \ldots, n$

$k = 0, 1, \ldots, n-1$, $\ell = 1, 2, \ldots, n-1$, for convenience.

/Substituting....

Substituting the value of the derivatives $\frac{\partial^{n-1} z}{\partial x^{n-1-i} \partial y^{i}}$, $i = 0,1,\cdots, n-1$, from (22), we obtain

$$\frac{\partial f_m}{\partial x} + \gamma_m \frac{\partial f_m}{\partial y} = \frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y} + \sum_{k=0}^{n-2} \sum_{i=0}^{k} \left[\left(\frac{\partial b_{ki}^{(m)}}{\partial x} + \gamma_m \frac{\partial b_{ki}^{(m)}}{\partial y} \right) + \left(b_{k-1,i}^{(m)} + \gamma_m b_{k-1,i-1}^{(m)} \right) \right]$$

$$\frac{\partial^k z}{\partial x^{k-i} \partial y^{i}} + \sum_{i=0}^{n-1} \sum_{p=1}^{k} \left(b_{n-2,i}^{(m)} + \gamma_m b_{n-2,i-1}^{(m)} \right) c_{ip} \left(f_p - \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b_{k\ell}^{(p)} \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} \right)$$

$$\ldots\ldots\ldots\ldots(26)$$

Again, substituting the expressions for $\frac{\partial z_m}{\partial x} + \gamma_m \frac{\partial z_m}{\partial y}$ as given by (24), we have after some modification,

$$\frac{\partial f_m}{\partial x} + \gamma_m \frac{\partial f_m}{\partial y} = a_0 + \sum_{k=0}^{n-2} \sum_{i=0}^{k} \left[\left(\frac{\partial b_{ki}^{(m)}}{\partial x} + \gamma_m \frac{\partial b_{ki}^{(m)}}{\partial y} \right) + \left(b_{k-1,i}^{(m)} + \gamma_m b_{k-1,i-1} \right) - a_{ki} \right] \frac{\partial^k z}{\partial x^{k-i} \partial y^{i}}$$

$$+ \sum_{i=0}^{n-1} \sum_{p=1}^{k} \left(b_{n-2,i}^{(m)} + \gamma_m b_{n-2,i-1}^{(m)} + \frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} \right.$$

$$\left. - a_{n-1,i} \right) c_{ip} \left(f_p - \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b_{k\ell}^{(p)} \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} \right)$$

$$= a_0 + \sum_{p=1}^{k} \left[\sum_{i=0}^{n-1} \left(b_{n-2,i}^{(m)} + \gamma_m b_{n-2,i-1}^{(m)} + \frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i} \right) c_{ip} \right] fp(x,y)$$

$$+ \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} \left(\frac{\partial b_{k\ell}^{(m)}}{\partial x} + \gamma_m \frac{\partial b_{k\ell}^{(m)}}{\partial y} \right) + \left(b_{k-1,\ell}^{(m)} + \gamma_m b_{k-1,\ell-1}^{(m)} \right) - a_{k\ell}$$

$$- \sum_{i=0}^{n-1} \sum_{p=0}^{n} \left(b_{n-2,i}^{(m)} + \gamma_m b_{n-2,i-1}^{(m)} + \frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i} \right)$$

$$c_{ip} b_{k\ell}^{(p)} \Big] \frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}} \quad \ldots\ldots\ldots(27)$$

It follows that if we define the functions $b_{ki}^{(m)}$ in such a way that the coefficients of $\frac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}}$ in (27) all vanish, then (27) becomes

$$\frac{\partial f_m}{\partial x} + \gamma_m \frac{\partial f_m}{\partial y} = b_{m1} f_1 + b_{m2} f_2 + \ldots + b_{mn} f_n + a_0, \; m = 1,2,\ldots,n.. \qquad \ldots\ldots\ldots(28)$$

where

$$b_{mk}(x,y) = \sum_{i=0}^{n-1} \left(b^{(m)}_{n-2,i} + \gamma_m b^{(m)}_{n-1,i-1} + \frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i} \right) c_{ik}$$

$$m,k = 1,2,\ldots,n \;\ldots\ldots\ldots(29)$$

The system of equations (28) will be called the resolvent of (1). It is of the type of equation (13) quoted in the introduction.

Referring to (27), we see that the condition that all the terms involving $\dfrac{\partial^k z}{\partial x^{k-\ell} \partial y^{\ell}}$ vanish will be satisfied if the functions $b^{(m)}_{k\ell}$, $m = 1,2,\ldots,n$, $k = 0,1,\ldots\, n-2$, $\ell = 0,\ldots,k$, are solutions of the system of partial differential equations

$$\frac{\partial b^{(m)}_{k\ell}}{\partial x} + \gamma_m \frac{\partial b^{(m)}_{k\ell}}{\partial y} = a_{k\ell} - \left(b^{(m)}_{k-1,\ell} + \gamma_m b^{(m)}_{k-1,\ell-1} \right)$$

$$+ \sum_{i=0}^{n-1} \sum_{p=1}^{k} c_{ip} \left(b^{(m)}_{n-2,i} + \gamma_m b^{(m)}_{n-2,i-1} + \frac{\partial a_i^{(m)}}{\partial x} + \gamma_m \frac{\partial a_i^{(m)}}{\partial y} - a_{n-1,i} \right) b^{(p)}_{k\ell} \quad ..(30)$$

with $b_{-1,0} = b_{-1,-1} = b_{0,-1} = 0$, by definition.

(30) will be called the auxiliary system. It will be seen that the expressions on its right hand side are quadratic functions of the dependent variables: it therefore constitutes a system of the type of (14). The number of equations in (30) is $\frac{1}{2}n^2(n-1)$.

Assume now that the values of z and of its first $(n-1)$ derivatives with respect to x are specified for a range of values of y, for $x = 0$,

$$z(0,y) = g_0(y), \left(\frac{\partial z}{\partial x}\right)_{x=0} = g_1(y), \ldots, \left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right)_{x=0} = g_{n-1}(y) \quad ..(31)$$

where $g_i(y)$, $i=0, \ldots, n-1$ can be differentiated $n-i-1$ times with respect to y. Having solved (30) for arbitrary but sufficiently regular initial conditions (see section 3, below), we can then obtain the initial conditions for the functions $f_m(x,y)$ from (20) and (31),

$$f_m(0,y) = g_{n-1}(y) + a_1^{(m)} \frac{\partial g_{n-2}}{\partial y} + \ldots + a_{n-1}^{(m)} \frac{\partial^{n-1} g_0}{\partial y^{n-1}} + \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} b^{(m)}_{k\ell} \frac{\partial^{\ell} g_{k-\ell}}{\partial y^{\ell}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$

/Having ......

Having solved (28) for these initial conditions, we may then look upon anyone of the equations (20) as a linear partial differential equation of order $(n-1)$ for z. All these equations are of hyperbolic type, the roots of their characteristic equations consisting of $(n-1)$ of the n roots of (3). By the successive application of this procedure, we finally obtain a first order equation for z, which is itself an example of (13).

It only remains for us to show that the determinant of (21),

$$D = \begin{vmatrix} 1 & a_1^{(1)} & \dots & a_{n-1}^{(1)} \\ \vdots & & & \\ 1 & a_1^{(n)} & \dots & a_{n-1}^{(n)} \end{vmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots(33)$$

does not vanish in the region under consideration.

We write $S_1(x_1, x_2, \dots, x_m)$, $S_2(x_1, x_2, \dots, x_m)$, ...,

$$S_m(x_1, x_2, \dots, x_m)$$

for the fundamental symmetrical functions of a set of variables $x_1, x_2, \dots, x_m$,

$$S_1(x_1, x_2, \dots, x_m) = \sum x_i,\ S_2(x_1, x_2, \dots, x_m) = \sum_{i \neq k} x_i x_k, \dots;$$

$$S_m(x_1, x_2, \dots, x_m) = x_1 x_2 \dots x_m \qquad \dots\dots(34)$$

The determinant D can then be expressed as

$$D = \begin{vmatrix} 1 & S_1(\gamma_2, \gamma_3, \dots, \gamma_n) & S_2(\gamma_2, \gamma_3, \dots, \gamma_n) & \dots & S_{n-1}(\gamma_2, \gamma_3, \dots, \gamma_n) \\ 1 & S_1(\gamma_1, \gamma_3, \dots, \gamma_n) & S_2(\gamma_1, \gamma_3, \dots, \gamma_n) & \dots & S_{n-1}(\gamma_1, \gamma_3, \dots, \gamma_n) \\ \vdots & \dots\dots & \dots\dots & & \dots\dots \\ 1 & S_1(\gamma_1, \gamma_2, \dots, \gamma_{n-1}) & S_2(\gamma_1, \gamma_2, \dots, \gamma_{n-1}) & \dots & S_{n-1}(\gamma_1, \gamma_2, \dots, \gamma_{n-1}) \end{vmatrix} \qquad \dots\dots(35)$$

Now

$$S_1(\gamma_1, \gamma_3, \dots, \gamma_n) - S_1(\gamma_2, \gamma_3, \dots, \gamma_n) = \gamma_1 - \gamma_2$$

$$S_2(\gamma_1, \gamma_3, \dots, \gamma_n) - S_2(\gamma_2, \gamma_3, \dots, \gamma_n) = (\gamma_1 - \gamma_2)\ S_1(\gamma_3, \dots, \gamma_n)$$

$$\vdots$$

$$S_{n-1}(\gamma_1, \gamma_3, \dots, \gamma_n) - S_{n-1}(\gamma_2, \gamma_3, \dots, \gamma_n) = (\gamma_1 - \gamma_2)\, S_{n-2}(\gamma_3, \dots, \gamma_n)$$

/contd. over...

$$\begin{array}{l}
\cdot \\ \cdot \\ \cdot \\
S_1(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) - S_1(\gamma_2, \gamma_3, \ldots, \gamma_n) = \gamma_1 - \gamma_n \\
S_2(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) - S_2(\gamma_2, \gamma_3, \ldots, \gamma_n) = (\gamma_1 - \gamma_n) \; S_1(\gamma_2, \ldots, \gamma_{n-1}) \\
\cdot \\ \cdot \\ \cdot \\
S_{n-1}(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) - S_{n-1}(\gamma_2, \gamma_3, \ldots, \gamma_n) = (\gamma_1 - \gamma_n) S_{n-2}(\gamma_2, \ldots, \gamma_{n-1})
\end{array}$$

....................(36)

Hence, subtracting the first row in (35) from the second,..., n^{th} row, we obtain

$$D = \begin{vmatrix} 1 & S_1(\gamma_2, \gamma_3, \ldots, \gamma_n) & S_2(\gamma_2, \gamma_3, \ldots, \gamma_n) & \cdots & S_{n-1}(\gamma_2, \gamma_3, \ldots, \gamma_n) \\ 0 & (\gamma_1 - \gamma_2) & (\gamma_1 - \gamma_2) S_1(\gamma_3, \ldots, \gamma_n) & \cdots & (\gamma_1 - \gamma_2) S_{n-2}(\gamma_3, \ldots, \gamma_n) \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & (\gamma_1 - \gamma_n) & (\gamma_1 - \gamma_n) S_1(\gamma_2, \ldots, \gamma_{n-1}) & . & (\gamma_1 - \gamma_n) S_{n-2}(\gamma_2, \ldots, \gamma_{n-1}) \end{vmatrix}$$

$$= (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3) \cdots (\gamma_1 - \gamma_n) \begin{vmatrix} 1 & S_1(\gamma_3, \ldots, \gamma_n) & \cdots & S_{n-2}(\gamma_3, \ldots, \gamma_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & S_1(\gamma_2, \ldots, \gamma_{n-1}) & & S_{n-2}(\gamma_2, \ldots, \gamma_{n-1}) \end{vmatrix}$$

..................(37)

/Regarding ....

Regarding D as a function of the variables $\gamma_1, \gamma_2, \ldots, \gamma_n$, $D = D_n(\gamma_1, \gamma, \ldots, \gamma_n)$, we see that (37) becomes

$$D_n(\gamma_1, \gamma_2, \ldots, \gamma_n) = (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3) \cdots (\gamma_1 - \gamma_n)\, D_{n-1}(\gamma_2, \ldots, \gamma_n) \qquad \ldots\ldots\ldots\ldots\ldots(38)$$

Also $D_2(\gamma_{n-1}, \gamma_n) = \gamma_{n-1} - \gamma_n$. Hence finally

$$D_n(\gamma_1, \gamma_2, \ldots, \gamma_n) = \prod_{i<k} (\gamma_i - \gamma_k) \qquad \ldots\ldots\ldots\ldots\ldots(39)$$

This shows that D^2 is simply the discriminant of the characteristic equation. Since $\gamma_i \neq \gamma_k$ for $i \neq k$, $D \neq 0$ as asserted.

3. Integration

We shall now discuss the integration of the system of equations

$$\frac{\partial f_i}{\partial x} + c_i \frac{\partial f_i}{\partial y} = F_i(x, y, f_1, f_2, \ldots, f_m), \quad i = 1, 2, \ldots, m \qquad \ldots.(14)$$

for given initial conditions

$$f_1(0,y) = h_1(y),\ f_2(0,y) = h_2(y), \ldots,\ f_m(0,y) = h_m(y) \quad \ldots\ldots(40)$$

It will be assumed throughout that the $c_i = c_i(x,y)$ possess first derivatives in the region under consideration. Other conditions will be laid down in the course of the analysis. With each one of the equations in (14), we associate a one-parametric family of characteristic curves $y = \phi_i(x)$ given by

$$\frac{dy}{dx} = c_i(c,y) \qquad i = 1, 2, \ldots, m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41).$$

We denote differentiation along a characteristic curve $y = \phi_i(x)$ by $\frac{D_i}{D_{ix}}$, so that for any arbitrary function $f(x,y)$, $\frac{D_{if}}{D_{ix}}$ is defined by

$$\frac{D_i f(x,y)}{D_{ix}} = \lim_{h \to 0} \frac{f(x+h,\ \phi_i(x+h)) - f(x, \phi_i(x))}{h}, \quad y = \phi_i(x) \qquad \ldots\ldots\ldots\ldots(42)$$

whenever that limit exists. Thus, if f possesses continuous first derivatives,

$$\frac{D_i f}{D_i x} = \frac{\partial f}{\partial x} + \frac{d\phi_i}{d\,x}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} + c_i(x,y)\frac{\partial f}{\partial y} \quad \text{.......................}(43)$$

Equations (14) can now be written as

$$\frac{D_i f_i}{D_i x} = F_i\ (x,y,f_1,f_2,\ldots,fm),\ i = 1,2,\ldots,m \quad \text{..............}(44)$$

Integrating between two values x' and x'' along a characteristic curve $\phi_i(x)$, we obtain

$$f_i\big(x'',\phi_i(x'')\big) - f_i(x',\phi_i(x'))$$
$$= \int_{x'}^{x''} F_i\ \Big(\xi\,,\phi_i(\xi),\ f_1\big(\xi,\phi_i(\xi)\big),\ldots,\ f_m\big(\xi,\phi_i(\xi)\big)\Big)\ d\xi$$
$$\text{...........}(45)$$

In particular, taking $x' = 0$ and writing x for x'', we have

$$f_i\ \big(x,\phi_i(x)\big) = f_i\ \big(0,\phi_i(0)\big)$$
$$+ \int_0^x F_i\ \Big(\xi,\phi_i(\xi),f_1\ \big(\xi,\phi_i(\xi)\big),\ldots,f_m\big(\xi,\phi_i(\xi)\big)\Big)\ d\xi$$
$$i = 1,2,\ldots,m \quad \text{........}(46)$$

where ϕ_i is any particular characteristic curve which meets the y axis at some point of the region under consideration.

Equations (46) suggest a method of successive approximation by which the functions $f_i(x,y)$ can be determined for given initial values. The functions $f_{i0}(x,y)$, $i = 1,2,\ldots,m$ are first defined by

$$f_{1,0}(x,y) = h_1\ \big(\phi_1(0)\big)\ ,\ f_{2,0}(x,y) = h_2\ \big(\phi_2(0)\big)\ ,\ldots,$$
$$f_{m,0}(x,y) = h_m\ \big(\phi_m(0)\big) \quad \text{............}(47)$$

where the characteristic curves in question pass through the point (x,y), $y = \phi_1(x) = \phi_2(x) = \ldots = \phi_m(x)$.

/The ....

The functions $f_{i\mu}(x,y)$, $i = 1,2,\dots,m$, $\mu \geq 1$ are then determined successively by

$$f_{i\mu}(x,y) = h_i\left(\phi_i(0)\right) + \int_0^x F_i\left(\xi, \phi_i(\xi), f_{1,\mu-1}\left(\xi,\phi_i(\xi)\right), \dots, f_{m,\mu-1}\left(\xi,\phi_i(\xi)\right)\right) d\xi \quad \dots\dots(48)$$

where the characteristic curves are chosen as above.

A set of conditions under which the functions $f_{i\mu}(x,y)$ can be constructed such that the limits $f_i(x,y) = \lim_{\mu\to\infty} f_{i\mu}(x,y)$ exist and form a solution of (14) for the initial values (40), will now be given.

It will be assumed that the functions $c_i(x,y)$ are defined and differentiablewith respect to x and y in a closed region R in the right half of the (x,y) plane $(x \geq 0)$. The region R will be supposed to include an interval $\langle a, b\rangle$ of the y-axis as part of its boundary, such that the functions $h_i(y)$ are defined and continuous in that interval. It will be assumed further that the functions $F_i\left(x,y,u_1,\dots,u_m\right)$ are continuous functions of $x,y,u_1,u_2,\dots,u_m$ for all (x,y) in R and for all u_i such that there exists a y, $a \leq y \leq b$ for which

$$|u_i - h_i(y)| \leq q \quad \dots\dots(49)$$

where q is a positive constant. For all such x,y and u_i the functions F_i will be supposed to satisfy Lipschitz conditions with constants $N_1, N_2, \dots, N_m$, viz.,

$$\left| F_i\left(x,y, u_1', u_2', \dots, u_m'\right) - F_i\left(x,y,u_1,u_2,\dots,u_m\right)\right| < N_1 | u_1' - u_1 | + N_2 | u_2' - u_2| + \dots + N_m | u_m' - u_m | \quad i = 1,2,\dots,m \quad \dots\dots(50)$$

where the (x,y) are in R, and the u_i and u_i' satisfy (49).

Since the $h_i(y)$ are continuous, it follows that the set S of points $\left(x,y,u_1,u_2,\dots,u_m\right)$ in $\left(x,y,u_1,u_2,\dots,u_m\right)$ space such that x,y is in R while the u_i satisfy (49), is closed. The F_i being continuous in this set are therefore bounded in it: there exists a constant M such that

$$\left| F_i\left(x,y,u_1,u_2,\dots u_m\right)\right| < M, \quad i = 1,2,\dots,m \quad \left(x,y,u_1,u_2,\dots,u_m\right) \text{ in } S. \quad \dots\dots(51)$$

/Let .....

Let R' be the set of points x_0, y_0 in R which are such that every continuous curve $y = \phi(x)$, $x \leq x_0$, $\phi(x_0) = y_0$, which consists of a finite number of segments of characteristic curves $\phi_i(x)$ in R, can reach the boundary of R only on $\langle a,b \rangle$ (except possibly also in (x_0,y_0)). This condition will be satisfied by all points (x_0,y_0), $x_0 > 0$ which are sufficiently close to any given interior point of $\langle a,b \rangle$. In the case of equations (28) and (30) it simply means that if we draw the characteristic curves of greatest and least slopes through (x_0,y_0) they will meet the boundary of R in the interval $\langle a,b \rangle$.

Let $d = q/M$, and let R'' be the subset of points (x,y) of R such that $x \leq d$. We denote by S'' the set of points $(x,y,u_1,u_2,\dots,u_m)$ in S such that (x,y) is in R'' while the u_i satisfy (49).

We shall show that under the stated conditions the construction (48) can be carried on indefinitely for all points in R''. In fact, for all points in S'', $|F_i| < M$ and so, by (48),

$$|f_i \quad (x,y) - h_i(\phi_i(0))| < Md = q, \quad y = \phi_i(x), \quad i = 1,2,\dots,m \qquad \dots\dots(52)$$

It follows that whenever (x,y) is in R'', $(x,y,f_{11},f_{21},\dots f_{m1})$ is in S'' since the $u_i = f_{i1}$ satisfy (49). We then deduce in the same way that $|f_{i2}(x,y) - h_i(\phi_i(0))| < q$ for all points in R'', etc. This shows that the fact that $(x,y,f_{1\mu},f_{2\mu}\dots f_{m\mu})$ is in S'' implies that $(x,y,f_{1,\mu+1}, f_{2,\mu+1},\dots,f_{m,\mu+1})$ also is in S'', and hence that the construction can be continued indefinitely.

Now we have, for all (x,y) in R'',

$$|f_{i1} - f_{i0}| = \left|\int_0^x F_i(\xi, \phi_i(\xi), f_{10}(\xi,\phi_i(\xi)),\dots,f_{m0}(\xi,\phi_i(\xi)))\, d\xi\right|$$
$$\leq Mx \qquad i = 1,2,\dots,m \qquad \dots\dots(53)$$

Also,

$$|f_{i2} - f_{i1}| = \left|\int_0^x F_i(\xi,\phi_i(\xi),f_{11}(\xi,\phi_i(\xi)) \quad \dots) - F_i(\xi,\phi_i(\xi),f_{10}(\xi,\phi_i(\xi)) \quad \dots)\, d\xi\right|$$

$$\leq \int_0^x (N_1 |f_{11} - f_{10}| + \dots + N_m |f_{m1} - f_{m0}|)\, d\xi$$

$$\leq (N_1 + N_2 + \dots + N_m) M \int_0^x \xi\, d\xi = MN \frac{x^2}{2} \qquad \dots\dots(54)$$

/where ......

where $N = N_1 + N_2 + \dots + N_m$.

Similarly,

$$|f_{i3} - f_{i2}| \leq M N^2 \frac{x^3}{3!} \quad \text{for points in } R''$$

and, in general,

$$|f_{i,\mu+1} - f_{i,\mu}| \leq MN^{\mu} \frac{x^{\mu+1}}{(\mu+1)!} \leq MN^{\mu} \frac{d^{\mu+1}}{(\mu+1)!},$$

$$\mu = 0,1,2,\dots, \qquad \dots\dots\dots\dots(55)$$

Since the series

$$Md + MN \frac{d^2}{2} + MN^2 \frac{d^3}{3!} + \dots + MN^{\mu} \frac{d^{\mu+1}}{(\mu+1)!} + \dots$$

converges, it follows that the series

$$f_{i0}(x,y) + \Big(f_{i1}(x,y) - f_{i0}(x,y)\Big) + \Big(f_{i2}(x,y) - f_{i1}(x,y)\Big)$$

$$+ \Big(f_{i3}(x,y) - f_{i2}(x,y)\Big) + \dots$$

$$i = 1,2,\dots,m \qquad \dots\dots\dots(56)$$

are uniformly convergent in R''. Calling the limits $f_i(x,y)$, we then have, taking into account (48),

$$f_i(x,y) = h_i\Big(\phi_i(0)\Big) + \int_0^x F_i\Big(\xi, \phi_i(\xi), f_1\big(\xi, \phi_i(\xi)\big), \dots\Big) d\xi$$

$$i = 1,2,\dots,m, \text{ where } \phi_i(x) = y \quad \dots(57).$$

(57) in turn implies

$$\frac{D_i f_i}{D_i x} = F_i\Big(x, y, f_1(x,y), \dots, f_m(x,y)\Big) \quad i = 1,2,\dots,m, \qquad \dots\dots(58)$$

everywhere in R'', and

$$f_i(0,y) = h_i(y), \; i = 1,2,\dots,m \qquad \dots\dots\dots\dots\dots\dots\dots\dots(59).$$

/The....

The above procedure has been developed in close analogy with Picard's method for the construction of the solution of a system of ordinary differential equations by successive approximation (ref.6). By applying Lindelöf's modification of Picard's method, we can show that the region R'' may be replaced by an alternative region R''_0 in the above demonstration. To define this region, let M_0 be the maximum of the functions $h_i(y)$ in $\langle a,b \rangle$. Evidently, M_0 also is the maximum of the functions $f_{i0}(x,y)$ in R'. We may then show successively, by a method similar to that adopted earlier, that

$$|f_{i\mu} - f_{i0}| < M_0 x + M_0 N \frac{x^2}{2} + \dots + M_0 N^{\mu} \frac{x^{\mu+1}}{(\mu+1)!} + \dots$$

or

$$|f_{i\mu} - f_{i0}| < \frac{M_0}{N} (\exp Nx-1) \quad i = 1,2,\dots,m\,,\ \mu = 1,2,\dots \qquad \dots\dots\dots(60)$$

Hence, so long as

$$\frac{M_0}{N} (\exp Nx-1) \leq q$$

i.e., so long as

$$x \leq \frac{1}{N} \log \left(1 + \frac{Nq}{M_0}\right) \qquad \dots\dots\dots\dots(61)$$

we may carry on the construction given by (48) indefinitely, and then continue the argument as before. We define the region R''_0 as the set of all points in R' whose abscissae satisfy (61).

If the functions $F_i(x,y,u_1,u_2,\dots,u_m)$ are polynomials of $u_1,u_2,\dots,u_m$ with coefficients depending on x and y, then the constant q can be taken to be arbitrarily large, so that (61) does not impose any effective condition at all. In that case then, $R''_0 = R'$.

We still have to consider whether $\dfrac{D_i f_i}{D_i x}$ in (58) can be replaced by $\dfrac{\partial f_i}{\partial x} + c_i(x,y) \dfrac{\partial f_i}{\partial y}$.

This is certainly not necessarily the case under the assumptions made so far. In fact, until now we have only assumed the $h_i(y)$ to be continuous, and even that assumption might have been relaxed still further without affecting the preceding developments in any essential way. However, in order to be able to replace

$$\frac{D_i f_i}{D_i x} \text{ by } \frac{\partial f_i}{\partial x} + c_i(x,y) \frac{\partial f_i}{\partial y}$$

we require a set of more stringent conditions.

/Accordingly.....

Accordingly, we shall assume that the functions $h_i(y)$ have continuous first derivatives in $\langle a,b\rangle$, and also that the derivatives $\frac{\partial F_i}{\partial x}, \frac{\partial F_i}{\partial y}, \frac{\partial F_i}{\partial u_k}$, $i, k = 1,2,\ldots,m$ exist and are continuous in S.

Fixing our attention on a specific value of x, x_0, we may regard the functions $y = \phi_i(x)$ passing through points x_0, y_0 in R' as functions of y_0,

$$y = \phi_i(x;y_0) \quad \text{satisfies} \quad y_0 = \phi_i(x_0;y_0) \quad i = 1,2,\ldots m \qquad \ldots\ldots\ldots(62)$$

It then follows from known results on ordinary differential equations that ϕ_i is a differentiable function of y_0 for all $0 \leq x \leq x_0$ and that the derivative is a continuous function of y_0.

To proceed, we require the following lemma.
Let a set of functions $g_{i\mu}(x,y)$, $i = 1,2,\ldots,m$, $= 1,2,\ldots$ be defined by

$$g_{1,0}(x,y) = p_1(x,y)\, k_1(\phi_1(x)),\ldots,\ g_{1m}(x,y) = p_m(x,y) k_m(\phi_m(x)) \qquad \ldots\ldots\ldots(63)$$

and by

$$g_{1\mu}(x,y) = p_i(x,y)\, k_i(\phi_i(x)) + \int_0^x G_{i\mu}(\xi, \phi_i(\xi), g_{1,\mu-1}(\xi, \phi_i(\xi)), \ldots$$
$$\ldots g_{m,\mu-1}(\xi,\phi_i(\xi)))\, d\xi$$
$$\text{for} \quad , \text{where } y = \phi_i(x). \qquad \ldots\ldots\ldots(64)$$

The functions $\phi_i(x)$ are supposed to be defined in a region R, as before. The functions $p_i(x,y)$ are supposed to be continuous in R while the $h_i(x,y)$ are continuous in $\langle a,b\rangle$ which forms part of the boundary of R. The functions $G_i(x,y,u_1,u_2,\ldots,u_m)$ are supposed to be continuous in a region S defined with the aid of condition (49) as before, they are supposed to be uniformly convergent in that region,

$$\lim_{\mu\to\infty} G_{i\mu} = G_i(x,y,u_1,u_2,\ldots,u_m) \text{ , say,} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(65)$$

and they are supposed to satisfy in S Lipschitz conditions with constants which are independent of the second suffices,

$$\left| G_{i\mu}(x,y,u_1',u_2',\ldots,u_m') - G_{i\mu}(x,y,u_1,u_2,\ldots,u_m) \right|$$
$$< N_1 |u_1' - u_1| + N_2 |u_2' - u_2| + \ldots + N_m |u_m' - u_m|$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(66).$$

/It ......

It follows from the continuity and uniform convergence of the $G_{i\mu}$ in S that these functions are also uniformly bounded in S. We define the subregion of R, R', which depends only on the characteristic curves $\phi_i(x)$, in the same way as before.

The required lemma then states that the functions $g_{i\mu}$ converge uniformly in a subregion R''_1 of R', $\lim_{\mu\to\infty} g_{i\mu}(x,y) = g_i(x,y)$ say, where R''_1 is defined after the manner of R'' or, alternatively, of R''_0. If the $G_{i\mu}$ are all polynomials of the variables u, R''_1 may be taken to coincide with R'.

The proof of the lemma is on the lines of the convergence proof for the functions $f_{i\mu}$.

Coming back to the main problem, we shall show that the functions $f_{i\mu}(x,y)$ defined by the procedure of (48) are differentiable with respect to y in R'' (or R''_0). In fact, we have from (47)

$$\frac{\partial f_{i,0}(x_0,y_0)}{\partial y_0} = \frac{\partial}{\partial y_0} h_i\left(\phi_i(0;\, y_0)\right) = h_i'\left(\phi_i\,(0;\, y_0)\right)\frac{\partial \phi_i}{\partial y_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(67).$$

Assume that we have already established differentiability with respect to y for all the functions f with second suffices $\mu - 1$. Then the derivative with respect to y of the integral in (48) exists and is continuous

$$\frac{\partial F_i}{\partial y_0} = \frac{\partial F_i}{\partial y}\cdot\frac{\partial \phi_i}{\partial y_0} + \frac{\partial F_i}{\partial u_1}\,\frac{\partial f_{1,\mu-1}}{\partial y_0} + \ldots + \frac{\partial F_i}{\partial u_m}\,\frac{\partial F_{m,\mu-1}}{\partial y_0} \qquad \ldots(68)$$

Hence

$$\frac{\partial f_{i\mu}(x_0,y_0)}{\partial y_0} = h_i'\left(\phi_i\,(0_i y_0)\right)\frac{\partial \phi_i}{\partial y_0} + \int_0^x \frac{\partial F_i}{\partial y_0}\, d\xi \qquad \ldots\ldots\ldots(69).$$

We now look upon the right hand side of (68) as a linear function of variables $u_i = \dfrac{\partial f_{i,\mu-1}}{\partial y_0}$ with coefficients which are known functions of x and y,

$$G_{i\mu} = \frac{\partial F_i}{\partial y}\cdot\frac{\partial \phi_i}{\partial y_0} + \frac{\partial F_i}{\partial u_1}\, u_1 + \ldots + \frac{\partial F_i}{\partial u_m}\, u_m \qquad \ldots\ldots\ldots\ldots\ldots(70)$$

/where.....

where $G_{i\mu} = G_{i\mu}(x,y,u_1,\ldots,u_m)$ depends on μ because it involves the functions $f_{i\mu-1}$, regarded however as known functions of x and y in R''. We also put $\frac{\partial\phi_i}{\partial y_0} = p_i(x_0, y_0)$, $i = 1,2,\ldots,m$ and $h_i'(y) = k_i(y)$. Then (67) is transformed into

$$g_{i0}(x,y) = p_i(x,y)\, k_i\left(\phi_i(0)\right) \quad , \qquad \ldots\ldots\ldots\ldots\ldots\ldots(71)$$

writing x,y for x_0,y_0. Also, (69) now becomes

$$g_{i\mu}(x,y) = p_i(x,y)\; k_i\left(\phi_i(0)\right) + \int_0^x G_{i\mu}\left(\xi, \phi_i(\xi),\; g_{1,\mu-1}\left(\xi, \phi_i(\xi)\right), \right.$$

$$\left. \ldots\; g_{m,\mu-1}\left(\xi, \phi_i(\xi)\right)\right) \; d\xi$$

$$i = 1,2,\ldots,m, \quad \mu = 1,2,.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(72)$$

(71) and (72) therefore are of the form of equations (63) and (64). In order to apply the lemma, we only have to verify that the requisite conditions are satisfied by the functions involved in (71) and (72). In particular, we see that the functions $G_{i\mu}$ are (inhomogeneous) linear functions of the u,

$$G_i = p^{(i)}_{0\mu} + p^{(i)}_{1\mu}\; u_1 + p^{(i)}_{2\mu}\; u_2 + \ldots + p^{(i)}_{m\mu}\; u_m$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(73)$$

where the functions $p^{(i)}_{k\mu}$ are continuous in R'' and are uniformly bounded in that region (since the $f_{i\mu}$ all satisfy (49)). It follows that there exist positive quantities P_k such that

$$\left| p^{(i)}_{k\mu} \right| < P_k \quad \text{in } R'' \quad i,k = 1,2,\ldots,m, \; \mu = 1,2,3,\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(74)$$

This in turn implies that the $G_{i\mu}$ satisfy a Lipschitz condition with constants which are independent of μ,

$$\left| G_{i\mu}(x,y,u_1', u_2',\ldots,u_m') - G_{i\mu}(x,y,u_1,u_2,\ldots,u_m) \right|$$

$$\leq P_1 \left| u_1' - u_1 \right| + P_2 \left| u_2' - u_2 \right| + \ldots + P_m \left| u_m' - u_m \right| \quad \ldots\ldots\ldots(75).$$

/By ....

By (61), we may then take R" as the R_1'' of the lemma, i.e., in R" the functions $g_{i\mu}(x,y)$) converge uniformly towards limits $g_i(x,y)$. Thus, the functions $\frac{\partial f_{i\mu}}{\partial y}$ converge uniformly in R", and this implies that the $f_i(x,y)$ are differentiable with respect to y in R", the value of the derivatives being $g_i(x,y)$. Moreover, these derivatives are continuous functions of x and y.

Next, we shall show that $\frac{\partial f_i}{\partial x}$ exists in R", $i = 1,2,\ldots,n$. For this purpose, consider the differences $f_i(x_0 + h, y_0) - f_i(x_0, y_0)$ at some specific point (x_0, y_0) in R". Let (x_0, y_1) be the point at which the characteristic curve $\phi_i(x)$ through $(x_0 + h, y_0)$ meets the straight line $x = x_0$. Then

$$f_i(x_0 + h, y_0) - f_i(x_0, y_0) = f_i(x_0 + h, y_0) - f_i(x_0, y_1)$$

$$+ f_i(x_0, y_1) - f_i(x_0, y_0)$$

Now by the mean value theorem, $f_i(x_0 + h, y_0) - f_i(x_0, y_1) = h\frac{D_i f_i}{D_i x}$ at some intermediate point of the characteristic curve in question, whose abscissa is ξ, say. Also $f_i(x_0, y_1) - f_i(x_0, y_0) =$

$$y_1 - y_0 \frac{\partial f_i(x_0, \eta)}{\partial \eta}$$

where η is a point between y_0 and y_1. Hence

$$\frac{f_i(x_0 + h, y_0) - f_i(x_0, y_0)}{h} = \frac{D_i f_i(x, \phi_i(x))}{D_i x}\bigg|_{x=\xi} + \frac{y_1 - y_0}{h} \frac{f_i(x_0, y)}{\partial y}\bigg|_{y=\eta} \quad \ldots\ldots\ldots(76)$$

But as h tends to 0, $\frac{y_1 - y_0}{h}$ tends to $-\frac{d\phi_i}{dx} = -c_i(x_0, y_0)$, and so, since $\frac{D_i f_i}{D_i x}$ is continuous along each characteristic, and $\frac{\partial f_i}{\partial y}$ is a continuous function of y, the limit of the expression on the left hand side of (76) exists, and equals

$$\frac{\partial f_i}{\partial x} = \frac{D_i f_i}{D_i x} - c_i(x,y)\frac{\partial f_i}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(77)$$

This proves that $\frac{D_i f_i}{D_i x}$ can indeed be replaced by $\frac{\partial f_i}{\partial x} + c_i(x,y)\frac{\partial f_i}{\partial y}$ in (58), showing that the $f_i(x,y)$ represent a solution of (14) in R".

/Finally.......

Finally, we shall establish the uniqueness of the solution obtained in this way. Referring to the constants $N_1, N_2, \dots, N_m$ in (50), we divide the half-plane $x \geq 0$ into strips of width $\frac{1}{2N}$, where $N = N_1 + N_2 + \dots + N_m$, by means of the straight lines $x = \frac{n}{2N}$, $n = 0, 1, 2, \dots$ This also divides R" into a number of closed regions $R_1, R_2, \dots$ corresponding to $0 \leq x \leq \frac{1}{2N}$, $\frac{1}{2N}$ x $\frac{2}{2N}$, etc. We are going to show, successively, that the solution is unique in every one of the regions R_i.

Assume that there exist two sets of solutions of (14) for the given initial conditions, f_i and f_i", say, and that at least one f_i differs from f_i' in R_1. Let ϵ be the greatest maximum attained by the differences $f_i - f_i'$ in R", and assume that this maximum is attained for a value $x = x_0$. Then

$$\left|f_i - f_i'\right| = \int_0^{x_0} \Big[F(\xi, \phi_i(\xi), f_1, f_2, \dots, f_m) - F(\xi, \phi_i(\xi), f_1', f_2', \dots, f_m')\Big] d\xi \Big|$$

$$\leq \int_0^{x_0} \Big(N_1 \max \left|f_i - f_1'\right| + N_2 \max \left|f_2 - f_2'\right| + \dots + N_m \max \left|f_m - f_m'\right|\Big) d\xi \quad \dots\dots\dots\dots(78)$$

where 'max' denotes maximum values in R_1. Hence

$$\left|f_i - f_i'\right| \leq \int_0^{x_0} \left(N_1 + N_2 + \dots + N_m\right) \epsilon d\xi = N \epsilon x_0 \leq \frac{N\epsilon}{2N} = \frac{\epsilon}{2} \quad \dots\dots\dots\dots(79)$$

This is contrary to assumption, and it follows that the functions f_i and f_i' are identical in R_1. In a similar way we show that they are identical in $R_2, \dots$, etc., and so in the entire region R".

This completes our proof of the existence and uniqueness of the solution of (14) for given initial conditions (40).

In addition to giving rise to a method of successive approximation, equations (44) also suggest a step-by-step method of integration, by which the functions $f_i(x,y)$ are determined approximately for $x + \delta x$ and for all y if they are known for x and for all y. The formulae of regression are

$$f_i\,(x + \delta x, y) = f_i(x, y - c_i\,\delta x) + F_i\,\delta x.$$

/However ......

However, in this form the method is not as yet very suitable for numerical work, since the formulae of regression require a knowledge of $f_i(x,y)$ for arbitrary values of y, whereas in practice it will be known only at a number of isolated points. Hence, in general, an additional operation of interpolation will be required at each step. The convergence of this process is a matter for further investigation.

Coming back to the solution of (1) through the intermediary of the resolvent (28) and of the auxiliary system (30), it will be seen that (30) satisfies the conditions of section 3, provided the coefficients $a_{n\ell}$, $\ell = 1,2,\dots,n$ and a_n possess continuous second derivatives with respect to both independent variables, while the other $a_{k\ell}$ possess at least continuous first derivatives. Under these circumstances, we only have to choose the initial values for the $b_{k\ell}^{(m)}$ in such a way that they have continuous derivatives in $\langle a,b \rangle$. One possible choice is $b_{k\ell}^{(m)}(0,y) = 0$ identically, but this is not the most skilful choice in all cases. However, if these conditions are satisfied in a region R as above, then the $b_{k\ell}^{(m)}$ also have first derivatives in the corresponding sub-region of R, R'. This in turn implies that the resolvent (28) also satisfies the conditions of section 3, provided (cf. (32)) the functions $g_{n-\ell}(y)$ possess continuous derivatives of order ℓ, $\ell = 1,2,\dots,n$. Finally, in that case, equations (20) regarded as equations for z, also satisfy corresponding conditions, so that the reduction of (1) can be continued to the end.

In conclusion, it may be said that the methods outlined in the present paper appear to have distinct prospects for numerical application. Their main advantage is that since integration is carried out along the characteristic curves, there is no possibility of a failure, such as may occur in the application of some lattice methods (see refs. 7,8). On the other hand the total number of integrations required for the solution of an equation of fairly high order is very large. However, even in these cases the procedure may still form a suitable basis for work on a modern calculating machine.

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