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A Linear State-Space Representation of Plane Poiseuille Flow for Control Design - A Tutorial

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Abstract: A method for the incorporation of wall transpiration into a model of linearised plane Poiseuille flow is presented, with the aim of producing a state-space model suitable for the development of feedback control of transition to turbulence in channel flow. The system state is observed via wall shear-stress measurements and controlled by wall transpiration. The streamwise discretisation in the linearised model is by Fourier series, and the wall-normal discretisation is by a Chebyshev polynomial basis, which is modified to conform to the control boundary conditions. The paper is intended as a tutorial on the addition of boundary control to a spectral model of a fluid continuum, to form a state-space model, as used in the emerging multidisciplinary field of flow control by means of MEMs (microelectrical machines). The ultimate aim of such flow control is the reduction of skin-friction drag on moving bodies.

Keywords: Flow control, linear systems, Poiseuille Flow, state space, Navier-Stokes equations

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1 INTRODUCTION

Hydrodynamic stability has been investigated for many years, see for example the seminal work by Orr (1907), and the phenomena of fluid flow transition from a laminar regime to a turbulent one, with associated increases in drag, heat transfer and mixing, is of great engineering importance. If transition can be suppressed, reductions in noise and propulsive power can be made, and if transition can be accelerated, heat transfer and mixing can be enhanced.

Passive and open-loop strategies have achieved limited results in practice. There is currently great interest in the fluids and control research communities regarding the feedback control of fluid-flow fields, as for example investigated by Joshi et al. (1997a), Bewley (2001) and Baramov et al. (2002b), using microelectrical machines technology (MEMS) as described by Ho and Tai (1998). If small perturbations in laminar flow can be effectively attenuated by feedback control, they will not grow rapidly and so instigate transition to turbulence. Theoretical studies and simulations, including those by Hogberg et al. (2003) and McKernan et al. (2004) are progressing , but feedback control of fluid flow using MEMs technology has not yet been achieved in practice.

For modern control theory to be used to design optimal robust control systems using standard software tools, such as those available in MATLAB, it is necessary to have a finite linear state-space model of the system dynamics of the form;-

$$\begin{aligned} \dot{\mathcal{X}} &= \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U} \\ \mathcal{Y} &= \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U} \end{aligned} \tag{1}$$

with inputs $\mathcal{U}$, state variables $\mathcal{X}$ and outputs $\mathcal{Y}$. Now the dynamics of fluid flow are governed by the Navier-Stokes equations which are infinite dimensional and non-linear. Furthermore, the sensing of the fluid disturbances and the effects of actuation on the fluid flow must also be incorporated. Such is the complexity of the equations that to date only the simplest of flows have been modelled for controller synthesis. Much work has addressed controller design for plane Poiseuille flow, the flow within infinite parallel planar boundaries, also known as channel flow.

This paper describes how the Navier-Stokes equations for plane Poiseuille flow can be approximated by a state-space model in the form of (1). The general scheme is shown in figure 1, where the base laminar Poiseuille flow with its parabolic velocity distribution flows between two parallel planar boundaries, and is subject to a flow disturbance. Wall shear stress measurements sense the disturbance and feed their output to a controller. The controller computes a suitable actuation that will stabilise and limit the disturbance, and the actuation signal is distributed to wall mounted actuators which blow or withdraw fluid in a direction normal to the wall, a process otherwise known as transpiration.

The approach described in this paper is based on the spectral code of Schmid and Henningson (2001, p487), originally developed for investigating Poiseuille flow hydrodynamic stability. There are a number of other ways of approaching the problem, for example as performed by Bewley and Liu (1998a), Joshi et al. (1997b), and Baramov et al. (2002a) but the method used here is free from spurious eigenvalues, has simple boundary conditions and is easily extensible beyond two dimensions.

This tutorial paper takes Schmid and Henningson's spectral code and presents a method of introducing wall transpiration and sensing to order to generate a state-space model, and demonstrates how the method is implemented for a 2-d case. The method performs a substitution on the system to form an inhomogeneous controlled system with homogeneous boundary conditions, with the use of Heinrich's basis functions for wall-normal discretisation which implicitly fulfill the homogeneous boundary conditions. The method is implemented as a state-space model in MATLAB.

In section 2 the flow equations linearised about plane Poiseuille flow are presented, and in section 3 they are reformulated into a velocity-vorticity form. In section 4, Schmid and Henningson's discretised model of the equations is reviewed. Section 5 develops a method of selecting state variables, applying the control boundary conditions, and measuring the system state. In section 6, the verification of the model is discussed, and section 7 briefly describes controller synthesis, simulation and implementation. Finally in section 8 concluding remarks are made.

The modelling goal in this paper is to produce a linear state-space control form representation of plane Poiseuille flow capable of modelling the very early stages of transition of the flow to turbulence, and the effects of wall transpiration inputs on these stages of transition, as measured by the behaviour of output wall shear-stresses.

2 FLOW EQUATIONS

2.1 The Incompressible Navier-Stokes and Continuity Equations

The Navier-Stokes equations represent the conservation of momentum of a Newtonian fluid. For an incompressible fluid with uniform density and viscosity, in the absence of body forces, in two-dimensional cartesian co-ordinates (x, y), the non-dimensionalised instantaneous velocity U, V and pressure P, are related by;-

$$\begin{aligned}\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y} &= -\frac{\partial P}{\partial x}+\frac{1}{R}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)\\ \frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y} &= -\frac{\partial P}{\partial y}+\frac{1}{R}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\end{aligned} \tag{2}$$

where R is the Reynolds number. The equations are non-linear, and coupled, and here their domain is a continuum in two dimensions in space, plus time. The continuity equation expresses conservation of mass, and, for an incompressible fluid in the absence of fluid sources and sinks, is;-

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{3}$$

The continuity equation represents a constraint on the fluid velocity spatial derivatives. These equations are accurately representative of the behaviour of many real fluids, such as air and water at normal pressures and temperatures, and low velocities.

The usual boundary conditions specify no slip i.e. zero velocity normal and tangential to solid boundaries. In due course, for control purposes, a time-dependent wall-normal fluid velocity is specified, representing wall transpiration.

2.2 Linearization About Plane Poiseuille Base Flow

Plane Poiseuille flow is the steady parallel flow between infinite planar parallel boundaries. Linearising the Navier-Stokes equations (2) for small perturbations (u, v, p) about non-dimensionalised plane Poiseuille flow $(\bar{U}, \bar{V}, \bar{P})$ (streamwise in x with walls normal to y at $y = \pm 1$, thus $\bar{U} = 1 - y^2, \bar{V} = 0, \forall t)$ results in;-

$$\begin{aligned}\frac{\partial u}{\partial t}+\bar{U}\frac{\partial u}{\partial x}+v\frac{\partial \bar{U}}{\partial y} &= -\frac{\partial p}{\partial x}+\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\\ \frac{\partial v}{\partial t}+\bar{U}\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y}+\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned} \tag{4}$$

When linearised, the continuity equation (3) becomes;-

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

and the boundary condition on velocities becomes no slip in terms of perturbations, i.e. $u = v = 0$ on the channel boundaries.

This linearization is the first in a series of approximations. Although the behaviour of a real fluid is non-linear, Trefethen et al. (1993) showed that a linear model can still capture critical unstable behaviour.

3 FORMULATION

3.1 System Formulations

The equations (4,5) form a system of 3 linear partial differential equations in three flow variables (u,v,p), in which the continuity equation contains no time derivatives, and acts as a spatial differential constraint. The constraint is enforced by the pressure acting as a Lagrange multiplier, keeping the velocity field divergence free. If algebraic equations are formulated from the system by discretisation, they are singular, as noted by Bewley (2001), since the continuity equation has no time derivatives, and so the matrix $\mathcal{L}$ in the descriptor form;-

$$\mathcal{L}\dot{\mathcal{X}} = \mathcal{A}^{\#}\mathcal{X}+\mathcal{B}^{\#}\mathcal{U} \tag{6}$$

where $\mathcal{A}^{\#}$ and $\mathcal{B}^{\#}$ are related forms of $\mathcal{A}$ and $\mathcal{B}$, cannot be inverted to produce the conventional state-space form (1). To proceed further, the system (4,5) is reformulated in terms of only one flow variable, a so-called divergence free basis in which continuity is implicitly enforced. The variables (u, v, p) are transformed to eliminate the continuity equation, and thus the differential constraint. This eliminates the algebraic constraints in the discretised form (6), reduces the order of $\mathcal{X}$ and $\mathcal{L}$ becomes non-singular. There are several possible formulations: vorticity-stream function, velocity-vorticity, and velocity-pressure, as described by Peyret (2002). The current work employs a velocity-vorticity formulation, similar to that of Bewley (2001), as it is convenient for the application of boundary conditions in the present simple geometry.

3.2 Velocity-Vorticity Formulation

Using the continuity perturbation equation (5), the pressure perturbation may be eliminated from the Navier-Stokes perturbation equations (4), to yield a 'wall-normal velocity' equation as described by Schmid and Henningson

(2001, p56);-

$$(\partial/\partial t)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\bar{U}\,(\partial/\partial x)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$
$$-\frac{\partial^2\bar{U}}{\partial y^2}\frac{\partial v}{\partial x}-\frac{1}{R}\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right)=0 \qquad (7)$$

This equation suffices to describe the 2-d problem described here, although a second equation, in vorticity, is required to completely describe a 3-d perturbation. It is noteworthy that the fourth order partial differential equation (PDE) (7) contains the time derivative of the Laplacian of velocity.

3.3 Boundary Conditions in Velocity-Vorticity Formulation

The no-slip boundary conditions in the velocity-vorticity formulation become;-

Dirichlet Boundary Conditions on v. The wall-normal velocity at the walls, $v(y=\pm 1)$, is zero in plane Poiseuille flow, and thus the boundary conditions are homogeneous. The introduction of wall transpiration will make the velocities non-zero and thus make the boundary conditions inhomogeneous.

Homogeneous Neumann Boundary Conditions on v. Substituting the zero streamwise velocity perturbation (u) at the wall into the continuity equation (5), the y derivative of wall-normal velocity perturbations is found to be zero at the walls;-

$$\left.\frac{\partial v}{\partial y}\right|_{y=\pm 1}=0 \qquad (8)$$

There are four boundary conditions for the fourth order velocity equation, thus forming a well posed mathematical problem.

4 SCHMID AND HENNINGSON'S MODEL

This section reviews the uncontrolled model of the linearised system of equations formulated by Schmid and Henningson (2001, p487), and describes the discretisation methods used in the streamwise and wall-normal directions.

4.1 Discretisation

Since the flow problem is a continuum and thus infinite dimensional in spatial coordinates, in order to work with a system with a finite number of states, the linearized system must be discretised in space, in the second approximation in the process of generating the plant model. Several methods of spatial discretisation of PDE's exist, e.g. finite difference, finite element, and finite volume methods. The system of differential equations is approximated and replaced by a system of algebraic equations.

Spectral discretisation methods are used here. Spectral methods belong to the class of weighted residual methods in which the weighted residual from evaluating the PDE using an approximate solution is set to zero. Spectral methods approximate the solution by a truncated series of orthogonal functions. Using a Fourier series for the orthogonal functions assumes a periodic solution, using a Chebyshev polynomial series assumes a non-periodic one. Here collocation is employed in the wall-normal direction, which involves setting the residual to zero at specific points, as compared to the Galerkin method which sets the average residual to zero, as described by Peyret (2002).

Bewley and Liu (1998a) and Aamo (2002) have presented state-space models of Poiseuille flow which use flowfield values at collocation points as state variables, by means of a cardinal function or interpolating basis. The approach here is to use a polynomial basis, in which the state variables are the spectral coefficients.

4.2 Streamwise Discretisation

The variation of the solution in the streamwise dimension is assumed to be periodic and approximated by terms from a truncated Fourier series. Rempfer (2003, p237) notes that this assumption leads to a temporally rather than spatially growing perturbation. Truncation of the series approximates infinite dimensional behaviour of the continuum by finite dimensional behaviour.

Thus solutions to the wall-normal velocity equation, (7), are approximated by;-

$$v(x,y,t)=Re\left(\sum_{n_{st}=0}^{N_{st}}\tilde{v}(y,t)e^{2\pi i n_{st}x/L_x}\right) \qquad (9)$$

where n_{st} is the streamwise harmonic number and L_x is a fundamental wavelength in the streamwise direction. N_{st} is finite and represents the truncation of the series.

The linearized equations decouple by harmonic number and thus it is possible to treat each number separately, so long as the boundary conditions and sensing are harmonic, as they are assumed here. For convenience, a wavenumber, $\alpha = 2\pi n_{st}/L_x$, is defined in cycles per 2π distance, and then the solution at each wavenumber is assumed to be of the form;-

$$v(x,y,t) = Re\left(\tilde{v}(y,t)e^{i\alpha x}\right) \tag{10}$$

where $\tilde{v}(y,t)$ is the wall-normal velocity perturbation Fourier coefficient, or more simply referred to as the velocity coefficient. This is complex and conveys the wall-normal (y) variation, and temporal (t) variation of v at real streamwise wavenumber α.

From the point of view of physical measurements, a Fourier transform is necessary to identify the degree to which each wavenumber is present. For example, $\tilde{v}$ at wavenumber α is given by;-

$$\tilde{v}(y,t) = \frac{1}{L_x}\int_a^{a+L_x} v(x,y,t)e^{i\alpha x}dx \tag{11}$$

which requires distributed sensing of v over x and subsequent integration. In practice a number of discrete sensors would be used, and a fast Fourier transform performed on their signals, although unusually in space rather than time.

Substituting the assumed solution (10) into the partial differential equation for wall-normal velocity (7), results in a wall-normal velocity perturbation Fourier space equation;-

$$\begin{aligned}\left(-\bar{U}\alpha^2 - \frac{\partial^2\bar{U}}{\partial y^2} - \frac{k^4}{iR\alpha}\right)\tilde{v} + \left(\bar{U} + \frac{2\alpha^2}{iR\alpha}\right)\frac{\partial^2\tilde{v}}{\partial y^2} \\ -\left(\frac{1}{iR\alpha}\right)\frac{\partial^4\tilde{v}}{\partial y^4} = \frac{i}{\alpha}\left(\frac{\partial^3\tilde{v}}{\partial y^2\partial t} - \alpha^2\frac{\partial\tilde{v}}{\partial t}\right)\end{aligned} \tag{12}$$

which will henceforth be referred to as the velocity equation. If an exponential time variation is assumed, the classical Orr-Sommerfeld equation results, with non-zero solutions at particular eigenvalues (temporal frequencies) for eigenfunctions $\tilde{v}(y)$.

The Orr-Sommerfeld modes have been accurately calculated by Orszag (1971). For $R > 5772$, the first mode is unstable, although linear stability is not the only control objective, as large transient energy growth brought about by the non-orthogonality of the system may invalidate assumptions about linearity, and ultimately cause transition to turbulence, as noted by Trefethen et al. (1993).

4.3 Discretization in the Wall-Normal Direction

Discretisation in the wall-normal direction is based on a Chebyshev polynomial series, which does not make an assumption of periodicity. Chebyshev polynomials, $\Gamma_n(y)$, of the first kind, are of the form (Råde and Westergren, 1999, p260);-

$$\Gamma_n(y) = \cos\left(n \arccos(y)\right) \tag{13}$$

where $-1 \leq y \leq 1$ (conveniently the same as the full non-dimensionalised channel height). So, for instance;-

$$\Gamma_0(y) = 1 \qquad \Gamma_1(y) = y \qquad \Gamma_2(y) = 2y^2 - 1$$

Chebyshev polynomials have many useful properties e.g. the minimax property of minimising the maximum error when approximating continuous functions (Fox and Parker, 1968), and are recommended by Boyd (2001, p10) where periodic boundary conditions are not applicable. Thus $\tilde{v}$ is approximated by a finite Chebyshev series;-

$$\tilde{v}(y,t) = \sum_{n=0}^{N} a_n(t)\Gamma_n(y) \tag{14}$$

where a_n are unknown spectral coefficients. The derivatives of the approximation with respect to y are obtained by differentiating the Chebyshev polynomials, for example, for the second derivative;-

$$\tilde{v}'' = \sum_{n=0}^{N} a_n(t)\Gamma_n''(y) \tag{15}$$

Schmid and Henningson (2001, p485) present recursion formulae for calculating these derivatives. Boyd (2001, p142) notes that the derivatives are known to become large at the ends of the range, for large N and for high order derivative.

The partial differential equations are expressed on a grid of $N+1$ Chebyshev-Gauss-Lobatto points (Chebyshev collocation points), y_k, where;-

$$y_k = \cos(\pi k/N), \quad k = 0, \ldots, N \tag{16}$$

in the technique known as collocation. This distribution of points is particularly favourable for spectral accuracy, and appropriately for the present problem, includes the boundary points (Peyret, 2002, p46).

Upon substitution of the Chebyshev series (14), the velocity equation (12) becomes;-

$$\begin{aligned}\left(-\bar{U}\alpha^2 - \frac{\partial^2\bar{U}}{\partial y^2} - \frac{k^4}{i\alpha R}\right)\sum_{n=0}^{N} a_n(t)\Gamma_n(y) \\ +\left(\bar{U} + 2\frac{\alpha^2}{i\alpha R}\right)\sum_{n=0}^{N} a_n(t)\Gamma_n{}''(y) \\ -\frac{1}{iR\alpha}\sum_{n=0}^{N} a_n(t)\Gamma_n{}''''(y) \\ = \frac{i}{\alpha}\left(-\alpha^2\sum_{n=0}^{N}\dot{a}_n(t)\Gamma_n(y) + \sum_{n=0}^{N}\dot{a}_n(t)\Gamma_n{}''(y)\right)\end{aligned} \tag{17}$$

4.4 Boundary Conditions

Schmid and Henningson discard equations at and next to the walls and replace them with algebraic equations representing the homogeneous Dirichlet and Neumann boundary conditions, in a technique known as boundary bordering. The technique introduces spurious eigenvalues, which may be moved to highly damped locations in the complex plane by a suitable choice of algebraic coefficients.

Here, the homogeneous Dirichlet boundary conditions $v(y=\pm 1)=0$ are implemented by the use of basis functions Γ^m which individually satisfy the conditions, i.e $v=\sum \Gamma_n^m(y)a_n$, where $\Gamma_n^m(y=\pm 1)=0$, as recommended by Boyd (2001, p114). The Neumann boundary conditions $v'(y=\pm 1)=0$ are also implemented directly by further requiring the basis functions to individually satisfy the Neumann conditions, i.e $\Gamma_n^m(y=\pm 1)'=0$. Thus the spurious eigenvalues introduced by the boundary bordering technique are avoided. Spurious eigenvalues introduced by local modification of the derivative matrix found by Bewley and Liu (1998a) and remedied by Aamo (2002, p59), are also avoided.

In this paper, the basis functions that satisfy the simultaneous homogeneous Dirichlet and Neumann boundary conditions are modified Chebyshev polynomials. The modification is that developed by Heinrichs (1989), multiplication by $(1-y^2)^2$;-

$$\tilde{v}=\sum a_n(1-y^2)^2\Gamma_n \triangleq \sum a_n\Gamma_n^m \triangleq \mathbf{D0}^m\,(a_n) \qquad (18)$$

This modification is similar to that employed by Hogberg et al. (2003), using the differentiation suite produced by Weideman and Reddy (2000, p499), although these sources work in terms of values at the collocation points, rather than spectral coefficients multiplying the Chebyshev polynomial series.

4.5 Evaluation at Collocation Points

After the evaluation of equation (17) at each of the collocation points y_k, and noting that $\bar{U}=1-y^2$, the equations may be assembled as;-

$$\mathbf{L}\left(\,\dot{a}_n\,\right) \;=\; \mathbf{A}\left(\,a_n\,\right) \qquad (19)$$

where

$$\begin{aligned}\mathbf{L} &= \left(-\alpha^2\mathbf{D0}^m+\mathbf{D2}^m\right) &(20)\\ \mathbf{A} &= \left(-\bar{\mathbf{U}}\alpha^3-\alpha\bar{\mathbf{U}}''-\frac{\mathbf{I}k^4}{iR}\right)\mathbf{D0}^m & \\ &+ \left(\alpha\bar{\mathbf{U}}+\frac{2\mathbf{I}\alpha^2}{iR}\right)\mathbf{D2}^m-\frac{\mathbf{D4}^m}{iR} &(21)\end{aligned}$$

Matrix $\mathbf{L}$ is the discrete form of the Laplacian operator $\left(\partial^2/\partial x^2+\partial^2/\partial y^2\right)$ which operates on the wall-normal velocity in (7). The elements of derivative matrices $\mathbf{Dn}^m$ are;-

$$D0_{kj}^m=\Gamma_j^m(y_k) \quad D2_{kj}^m=\Gamma_j^m(y_k)'' \quad D4_{kj}^m=\Gamma_j^m(y_k)'''' \qquad (22)$$

and diagonal base flow matrices $\bar{\mathbf{U}},\bar{\mathbf{U}}',\bar{\mathbf{U}}''$ are $diag(1-y_k^2), diag(-2y_k)$ and $diag(-2)$ respectively.

4.6 Final Schmid and Henningson Form

As Schmid and Henningson solve the Orr-Sommerfeld equation, they assume a time dependence $e^{-i\omega t}$ for coefficients a_n and arrive at a final form;-

$$-i\omega\mathbf{L}\left(\,a_n\,\right) \;=\; \mathbf{A}\left(\,a_n\,\right) \qquad (23)$$

The solution of this generalised eigensystem provides highly accurate eigenvalues ω and eigenfunctions (a_n) for periodic linearised plane Poiseuille flow, in the absence of any wall transpiration.

5 STATE-SPACE REPRESENTATION

This section identifies state, control and measurement variables, and proceeds to manipulate the equations prior to any assumption of time behaviour (19) into state-space form by the introduction of wall transpiration and wall sensing.

5.1 Selection of State Variables

State variables may be identified from the system of equations in many ways. A popular technique is to use the values of velocity at the collocation points, i.e. $\tilde{v}_k=\sum\Gamma_n(y_k)a_n$. The selection of state variables considered here is simply the spectral coefficients a_n which multiply Chebyshev polynomials, as used by Schmid and Henningson (2001). Although less intuitive than the collocation point values, the coefficients multiplying Chebyshev polynomials may be easily reduced in number when required, e.g., when matrices have to be kept square following the removal of equations, the variables multiplying the highest polynomials may be discarded with least consequences, due to the fast convergence of Chebyshev series (Fox and Parker, 1968, p25).

5.2 Selection of Control and Measurement Variables

Control of the fluid flow may be exercised by a number of means: electromagnetic body forces as investigated by Baker and Christofides (2002), variation of fluid viscosity by heating as by Hu and Bau (1994), or wall transpiration, as by Joshi et al. (1997b). Transpiration involves suction and blowing of fluid through the wall with zero net mass flow, and can be applied regardless of the fluid physical properties.

The control variables in the present work are related to the transpiration i.e. the wall-normal velocity perturbation values on the upper and lower walls, $\tilde{q}_u = \tilde{v}(y=1)$ and $\tilde{q}_l = \tilde{v}(y=-1)$. As these are Fourier coefficients, physically they correspond to sinusoidal distributions of transpiration velocity in the streamwise direction, at the wavenumber selected. Since the distributions are sinusoidal, zero-net mass transpiration is automatically enforced.

The state-space model requires fluid flow measurements, and the least intrusive at reasonable cost are those of wall shear-stresses. Thus the measurement variables in the present work are the wall shear-stress Fourier co-efficients on the upper and lower walls.

5.3 Introduction of Wall Transpiration

Without transpiration the linear differential equation (12) is homogeneous and thus may be expressed in the form $F(\tilde{v}) = 0$, and has homogeneous Dirichlet and Neumann boundary conditions;-

$$\tilde{v}(y=\pm 1) = 0, \tilde{v}'(y=\pm 1) = 0 \tag{24}$$

Non-zero wall-normal velocity introduces inhomogeneous Dirichlet boundary conditions into the homogeneous differential equations:

$$F(\tilde{v}) = 0, \tilde{v}(y=1) = \tilde{q}_u, \tilde{v}(y=-1) = \tilde{q}_l, \tilde{v}'(y=\pm 1) = 0 \tag{25}$$

where $\tilde{q}_u$ is the upper wall transpiration fluid wall-normal velocity, and $\tilde{q}_l$ the lower. Since it is linear, this homogeneous equation with inhomogeneous boundary conditions can be transformed to an inhomogeneous equation with homogeneous boundary conditions, by a suitable change of variable as described by Boyd (2001, p112). Defining;-

$$\tilde{v} = \tilde{v}_h + f_u \tilde{q}_u + f_l \tilde{q}_l \tag{26}$$

then providing;-

$$\begin{aligned} f_u(1) = f_l(-1) &= 1 \\ f_u(-1) = f_l(1) &= 0 \\ f_u'(\pm 1) = f_l'(\pm 1) &= 0 \end{aligned}$$

the equation and boundary conditions become;-

$$F(\tilde{v}_h) = -F(f_u \tilde{q}_u + f_l \tilde{q}_l), \tilde{v}_h(y=\pm 1) = 0, \tilde{v}_h'(y=\pm 1) = 0 \tag{27}$$

Joshi et al. (1997b) employed this method upon a stream function formulation, and Hogberg et al. (2003) on a velocity-vorticity formulation based on velocity values at collocation points.

Polynomials that satisfy the conditions for f_u and f_l are;-

$$f_u = \frac{(-y^3 + 3y + 2)}{4} \qquad f_l = \frac{(y^3 - 3y + 2)}{4} \tag{28}$$

These polynomials are slightly simpler than those used previously by Joshi.

5.4 Resulting System Equation

After substitution of the modified Chebyshev series (14) into the transformation (26) as $\tilde{v}_h$, and the application of the transformation to (12), and evaluation of that equations at the collocation points y_k, the equations may be assembled as;-

$$\begin{aligned} \mathbf{L}\left(\ \dot{a}_n\ \right) + i\left(-\alpha^2 \mathbf{f}_u + \mathbf{f}_u'', -\alpha^2 \mathbf{f}_l + \mathbf{f}_l''\right)\left(\begin{array}{c} \dot{\tilde{q}}_u \\ \dot{\tilde{q}}_l \end{array}\right) = \\ \left[\left(-\bar{\mathbf{U}}\alpha^3 - \alpha\bar{\mathbf{U}}'' - \frac{\mathbf{I}k^4}{iR}\right)[\mathbf{f}_u, \mathbf{f}_l] + \right. \\ \left.\left(\alpha\bar{\mathbf{U}} + \frac{2\alpha^2\mathbf{I}}{iR}\right)[\mathbf{f}_u'', \mathbf{f}_l'']\right]\left(\begin{array}{c} \tilde{q}_u \\ \tilde{q}_l \end{array}\right) \\ +\mathbf{A}\left(\ a_n\ \right) \end{aligned} \tag{29}$$

The elements of $(N+1) \times 1$ vectors $\mathbf{f}_u, \mathbf{f}_l, \mathbf{f}_u', \mathbf{f}_l', \mathbf{f}_u'', \mathbf{f}_l''$ are the values or derivatives of f_u and f_l, for example;-

$$\begin{aligned} f_{u_k} &= f_u(y_k) \\ f_{u_k}' &= f_u(y_k)' \end{aligned} \tag{30}$$

5.5 Measurement Equation

The wall sensing measurements are the non-dimensionalised wall shear stresses;-

$$\tau_{yx} = \frac{1}{R}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{31}$$

Now the values of v at the walls are set by the controller, and thus $\partial v/\partial x$ is known, and the measurement vector in Fourier space at a particular wavenumber can be selected as in Bewley (1999);-

$$\begin{pmatrix} \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_0} \\ \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_N} \end{pmatrix} \tag{32}$$

Since $\partial \tilde{u}/\partial y$ is $(i/\alpha)\partial^2 \tilde{v}/\partial y^2$ from the continuity equation (5);-

$$\begin{pmatrix} \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_0} \\ \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_N} \end{pmatrix} = \frac{i}{\alpha R}\begin{pmatrix} \frac{\partial^2 \tilde{v}}{\partial y^2}\Big|_{y_0} \\ \frac{\partial^2 \tilde{v}}{\partial y^2}\Big|_{y_N} \end{pmatrix} \tag{33}$$

Substituting the discretised form of $\tilde{v}$ (18);-

$$\begin{aligned} \begin{pmatrix} \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_0} \\ \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_N} \end{pmatrix} &= \frac{i}{\alpha R}(\mathbf{D2}^m)_{\text{rows } 0,N} \begin{pmatrix} a_n \end{pmatrix} \\ &+ \frac{3}{2R}\frac{i}{\alpha}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q_u \\ q_l \end{pmatrix} \end{aligned} \tag{34}$$

In a real system, the y derivatives of u would be calculated from τ_{yx} (31), and the derivatives of $\tilde{u}$ would be calculated from these by a Fourier transform similar to (11), at the appropriate wavenumber.

5.6 Resulting equations

Thus the system equation is;-

$$\mathbf{L}\begin{pmatrix} \dot{a}_n \end{pmatrix} + \mathbf{B2}\begin{pmatrix} \dot{q}_u \\ \dot{q}_l \end{pmatrix} = \mathbf{A}\begin{pmatrix} a_n \end{pmatrix} + \mathbf{B1}\begin{pmatrix} q_u \\ q_l \end{pmatrix} \tag{35}$$

where $\mathbf{L}$ and $\mathbf{A}$ are as previously (20,21), and;-

$$\mathbf{B2} = i\left(-\alpha^2\mathbf{f}_u + \mathbf{f}_u'', -\alpha^2\mathbf{f}_l + \mathbf{f}_l''\right) \tag{36}$$

$$\begin{aligned} \mathbf{B1} &= \left(-\alpha^3\bar{\mathbf{U}} - \alpha\bar{\mathbf{U}}'' - \frac{\mathbf{I}k^4}{iR}\right)(\mathbf{f}_u, \mathbf{f}_l) \\ &+ \left(\alpha\bar{\mathbf{U}} + \frac{2\alpha^2\mathbf{I}}{iR}\right)(\mathbf{f}_u'', \mathbf{f}_l'') \end{aligned} \tag{37}$$

The measurement equation is;-

$$\begin{pmatrix} \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_0} \\ \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_N} \end{pmatrix} = \mathbf{C}\begin{pmatrix} a_n \end{pmatrix} + \mathbf{D}\begin{pmatrix} q_u \\ q_l \end{pmatrix} \tag{38}$$

where;-

$$\mathbf{C} = \frac{i}{\alpha R}(\mathbf{D2}^m)_{\text{rows } 0,N} \tag{39}$$

$$\mathbf{D} = \frac{3}{2R}\frac{i}{\alpha}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{40}$$

The system equations at the walls are discarded, since there the velocity is set by the control variables and thus is known. In order to keep the matrices $\mathbf{A}$ and $\mathbf{L}$ square, the highest two Chebyshev polynomial co-efficients (a_{N-1}, a_N), are discarded also. This has minimal effect on the system because of the high rate of convergence of Chebyshev series.

5.7 State-Space Form

The system equation so far (35) is not in the standard form since it contains as control inputs both $\tilde{q}_u, \tilde{q}_l$ and $\dot{\tilde{q}}_u, \dot{\tilde{q}}_l$, which are not independent. This can be remedied by reclassifying $\tilde{q}_u, \tilde{q}_l$ as states and controlling purely by $\dot{\tilde{q}}_u, \dot{\tilde{q}}_l$, as suggested by Hogberg et al. (2003). This modification also results in a zero direct transmission matrix, which is necessary for Linear Quadratic Gaussian optimal control (Skogestad and Postlethwaite, 1996, p353).

The matrices of the conventional state-space form

$$\begin{aligned} \dot{\mathcal{X}} &= \mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{U} \\ \mathcal{Y} &= \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U} \end{aligned}$$

can be extracted as follows. The state matrix $\mathcal{A}$ is given by;-

$$\mathcal{A} = \begin{pmatrix} (\mathbf{L}^{-1}\mathbf{A}) & (\mathbf{L}^{-1}\mathbf{B1}) \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{41}$$

and the state variables $\mathcal{X}$ are;-

$$\mathcal{X} = \begin{pmatrix} a_n \\ \tilde{q}_u \\ \tilde{q}_l \end{pmatrix} \tag{42}$$

The input matrix $\mathcal{B}$ is given by;-

$$\mathcal{B} = \begin{pmatrix} \mathbf{L}^{-1}\mathbf{B2} \\ \mathbf{I} \end{pmatrix} \tag{43}$$

and the control vector $\mathcal{U}$ is;-

$$\mathcal{U} = \begin{pmatrix} \dot{\tilde{q}}_u \\ \dot{\tilde{q}}_l \end{pmatrix} \tag{44}$$

The output matrix $\mathcal{C}$ is given by;-

$$\mathcal{C} = \begin{pmatrix} \mathbf{C} & \mathbf{D} \end{pmatrix} \tag{45}$$

The direct transmission matrix $\mathcal{D}$ is 0_{2x2}, and the measurement vector $\mathcal{Y}$ is;-

$$\mathcal{Y} = \begin{pmatrix} \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_0} \\ \frac{1}{R}\frac{\partial \tilde{u}}{\partial y}\Big|_{y_N} \end{pmatrix} \tag{46}$$

6 MODEL VERIFICATION

The state-space system was implemented in MATLAB version 5.3. Detailed results were obtained and are reported by McKernan (2006). Comparing the eigensystems of the uncontrolled Schmid and Henningson form;-

$$\left(\dot{a}_n \right) = \mathbf{L}^{-1}\mathbf{A} \left(a_n \right) \tag{47}$$

with the controlled form (with control inputs $\dot{q}_u, \dot{q}_l$ zero),

$$\mathcal{A} = \begin{pmatrix} (\mathbf{L}^{-1}\mathbf{A}), & (\mathbf{L}^{-1}\mathbf{B1}) \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{48}$$

it is apparent that the state-space form has the same eigenvalues and eigenvectors as the uncontrolled Schmid and Henningson form, with the addition of two new zero eigenvalues. These eigenvalues have associated eigenvectors representing steady state transpiration at each wall. Without these new eigenvectors transpiration could not occur, since the Chebyshev basis used for the uncontrolled form is homogeneous at the walls.

Figure 2 shows the wall-normal perturbation velocity (v) flow field associated with one of the zero eigenvalues in the state-space model, for discretisation $N = 100$. Sinusoidal transpiration is clearly evident from the upper wall ($y = 1$), with the usual no-slip boundary condition at the lower wall.

The next 32 eigenvalues of the state-space system lie within one unit in the least significant digit of the published values for the Orr-Sommerfeld equations found by Orszag (1971), if the further eigenvalue found by Dongarra et al. (1996) is included in Orszag's sequence. Thus the uncontrolled dynamics has not been disrupted by the addition of the capability of wall transpiration. Furthermore, use of the ordinal difference proposed by Boyd (2001, p138) shows that no spurious eigenvalues have been introduced, as may occur in the discretised flow models of Bewley and Liu (1998a).

7 CONTROLLER DESIGN

The model has been used for the synthesis and simulation of optimal linear quadratic controllers (LQR) and linear quadratic Gaussian estimators (LQE), for $\alpha = 1, R = 10^4$, as described by McKernan et al. (2004) and McKernan et al. (2006). The controller state weights are chosen to minimise the sum of the flow perturbation kinetic energy density and controller effort. Minimisation of the perturbation kinetic energy density is appropriate as it is related to the degree of flow non-linearity and susceptibility to transition to turbulence. Different process noise weights are investigated in order to improve state estimator convergence.

The open- and closed-loop behaviour of the model compares very well with an independent finite volume computational fluid dynamics simulation, for linear sized perturbations. Whereas a finite volume simulation takes of the order of days, the linear model described here performs a simulation in a few minutes, due to absence of the non-linear terms and single wavenumber discretisation in the streamwise direction. Similarly, closed-loop finite volume simulations are not appreciably slowed down by the insertion of the linear controller into the code, as compared the execution time of open-loop finite volume simulations.

Linear matrix inequality (LMI) methods have been used to synthesize controllers which minimise bounds on the transient energy growth and control effort, by McKernan (2006) and Whidborne et al. (2006). H_∞ methods may also be used in order to synthesize robust controllers, as performed by Bewley and Liu (1998b) and Baramov et al. (2002b).

Practical implementations of the controller require relaxation of the assumption of streamwise periodicity, and reductions of the controller order, for example as described by Baramov et al. (2004) using a finite difference method and panel actuation, and also await developments in MEMs actuators. The actuators required must be sufficiently small to allow fine enough discretisation to control disturbances, and to be embedded in the walls, and yet of sufficient capacity to provide the required transpiration.

8 CONCLUDING REMARKS

A tutorial has been presented on the introduction of wall transpiration into a 2-d model of hydrodynamic stability, in order to produce a state-space system suitable for the design of optimal controllers. Many and detailed variations of the model are possible, as the references show, but it is hoped that this tutorial demonstrates the salient steps required. Extension to three dimensions involves the addition of a second equation, in the formulation adopted here, in vorticity.

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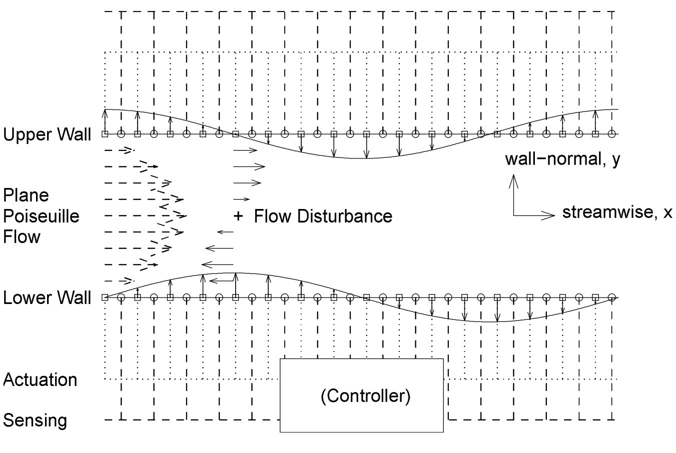

Figure 1:
Plane Poiseuille flow with wall transpiration and sensing

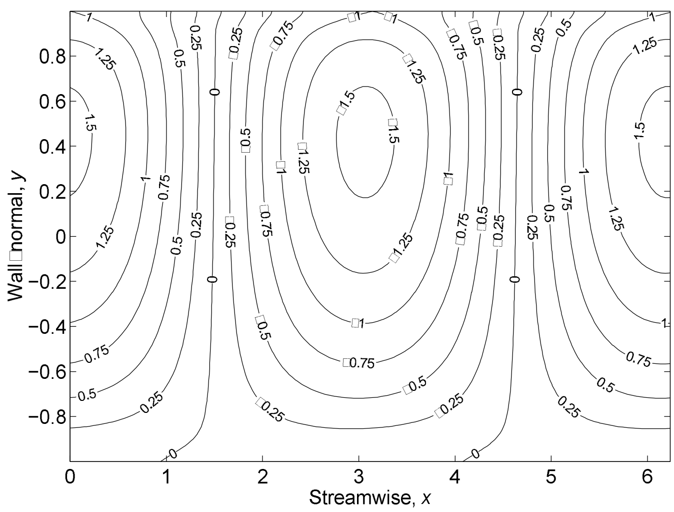

Figure 2:
Contours of the wall-normal velocity associated with a zero value eigenvalue of the wall transpiration state-space model, representing unit sinusoidal transpiration from the upper wall at $y = 1$ ($N = 100, \alpha = 1, Re = 10^4$)